Exercise 2. Use the Intermediate Value Theorem to find an interval of length one that contains a root of the equation. (a) $x^5 + x = 1$ (b) $\sin x = 6x + 5$ (c) $\ln x + x^2 = 3$.

Solution, part (a). Let $f_1(x) = x^5 + x - 1$. Since $f_1(0) = -1$ and $f_1(1) = 1$, $[0, 1]$ is a suitable interval.

Solution, part (b). Let $f_2 = \sin x - 6x - 5$. Since $f_2(-1) \approx 0.15853$ and $f_2(0) = -5$, $[-1, 0]$ is a suitable interval.

Solution, part (c). Let $f_3 = \ln x + x^2 - 3$. Since $f_3(1) = -2$ and $f_3(2) \approx 1.6931$, $[1, 2]$ is a suitable interval.

Exercise 4. Consider the equations in Exercise 2. Apply two steps of the Bisection Method to find an approximate root within 1/8 of the true root.

Solution, part (a). We will organize our work in the tabular form shown in the textbook:

k	a_k	$f(a_k)$	c_k	$f(c_k)$	b_k	$f(b_k)$
0	0.000	$-$	0.500	$-$	1.000	$+$
1	0.500	$-$	0.750	$-$	1.000	$+$
2	0.750	$-$	0.875		1.000	$+$

After two steps, our best estimate for the root is 0.875.

Solution, part (b). Similar actions are needed here:

k	a_k	$f(a_k)$	c_k	$f(c_k)$	b_k	$f(b_k)$
0	-1.000	$+$	-0.500	$-$	0.000	$-$
1	-1.000	$+$	-0.750	$-$	-0.500	$-$
2	-1.000	$+$	-0.875		-0.750	$-$

After two steps, our best estimate for the root is -0.875.

Solution, part (c). It's more of the same:

k	a_k	$f(a_k)$	c_k	$f(c_k)$	b_k	$f(b_k)$
0	1.000	$-$	1.500	$-$	2.000	$+$
1	1.500	$-$	1.750	$+$	2.000	$+$
2	1.500	$-$	1.625		1.750	$+$

After two steps, our best estimate for the root is 1.625.

Exercise 5. Consider the equation $x^4 = x^3 + 10$.

(a) Find an interval $[a, b]$ of length one inside which the equation has a solution.

(b) Starting with $[a, b]$, how many steps of the Bisection Method are required to calculate the solution within 10^{-10}? Answer with an integer.

Solution, part (a). Let $f(x) = x^4 - x^3 - 10$ and search for a root of $f(x) = 0$. $f(2) = -2$ and $f(3) = 44$, so $[2, 3]$ is an interval of length one which traps the root.

Solution, part (b). We know that

$$|x_c - r| < \frac{b-a}{2^{n+1}}$$

after n steps. Since our interval has length one, we must solve the inequality

$$\frac{1}{2^{n+1}} \leq 10^{-10},$$

looking for the smallest, whole number n. Here are the algebraic details:

$$\begin{aligned}
\frac{1}{2^{n+1}} &\leq 10^{-10} \\
2^{n+1} &\geq 10^{10} \\
(n+1)\log 2 &\geq 10 \log 10 \\
n+1 &\geq \frac{10 \log 10}{\log 2} \\
n &\geq \frac{10 \log 10}{\log 2} - 1 \\
&\approx 32.219
\end{aligned}$$

Therefore, performing $n = 33$ steps guarantees the error tolerance of 10^{-10}.