Approximating π with Machin's Formula

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March 19, 2014

Mathematical Background

To obtain an approximation for π, we will use two basic facts. First, a Taylor series expansion for arctangent:

$$\begin{aligned}\arctan x &= \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \\ &= \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+1}}{2j+1}\end{aligned}$$

Second, we use Machin's formula[1], which may appear somewhat mysterious:

$$\pi = 16 \arctan \frac{1}{5} - 4 \arctan \frac{1}{239}$$

Taylor series should be familiar to you from your study of calculus. Machin's formula is probably much less familiar to you, and can be derived with the use of trigonometric identities and algebraic manipulation. We omit those details here, as we plan to focus on the computational aspects of approximating π.

The Taylor series shown above involves an *infinite* number of terms, but we can only sum a *finite* number of terms with a computer. To more accurately describe the computations to be carried out, we introduce the following convention: let a_m be the partial sum consisting of the first m terms of the series given above. Thus,

$$\begin{aligned}a_1 &= \frac{x}{1} && j \text{ runs from 0 to 0} \\ a_2 &= \frac{x}{1} - \frac{x^3}{3} && j \text{ runs from 0 to 1} \\ a_3 &= \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} && j \text{ runs from 0 to 2} \\ a_4 &= \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} && j \text{ runs from 0 to 3} \\ &\vdots\end{aligned}$$

In general, observe that a_{k+1} is the partial sum with $k+1$ terms; in the summation, j runs from 0 to k, giving

$$a_{k+1} = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + \frac{(-1)^k x^{2k+1}}{2k+1}$$

[1] Developed by John Machin(1686–1751), an English astronomer and mathematician.

Here's another way to look at this collection of partial sums. Rather than think of each one as an unrelated summation of terms, we can make use of the similarity of each sum with the one that preceded it.

$$\begin{aligned} a_1 &= \frac{x}{1} \\ a_2 &= a_1 - \frac{x^3}{3} \\ a_3 &= a_2 + \frac{x^5}{5} \\ a_4 &= a_3 - \frac{x^7}{7} \\ &\vdots \end{aligned}$$

With the exception of a_1, each partial sum can be thought of as a sum consisting of "all but the last term" followed by "the last term." In general, for $k \geq 1$, we have:

$$a_{k+1} = a_k + \frac{(-1)^k x^{2k+1}}{2k+1}$$

Coding Details

With this background, we can now write a few lines of MATLAB which will compute ten partial sums, a_1 through a_{10}, for the arctangent at $x = 1/5$:

```
% Number of desired partial sums
n = 10;

% Where to evaluate arctan
x = 1/5;

% First partial sum
a(1) = x;

% Remaining partial sums
for k = 1:n-1
    a(k + 1) = a(k) + (-1)^k * x^(2*k+1)/(2*k+1);
end
```

For the tenth partial sum, we get $a_{10} = 0.197395559849881$, which is remarkably accurate considering that we used just 10 terms of the Taylor series.

To implement Machin's idea, we make a few simple modifications to this code. We now evaluate the arctangent function at two places: $x_A = \frac{1}{5}$ and $x_B = \frac{1}{239}$, saving partial sums in the vectors a and b. For a given pair a_i and b_i, we can compute an approximation to π using the Machin formula. The coding details are shown in Figure 1.

Generating Sensible Output

In most situations, we want to produce output which is clearly identified and has been neatly formatted. For tabulated data, this means we should have column headings and within each column, data should be formatted appropriately — integers should appear as integers and floating

```
% Arguments for atan()
xA = 1/5;
xB = 1/239;

% Total number of desired approximations
n = 10;

% atan approximations for xA and xB using just one term
a(1) = xA;
b(1) = xB;

% ...and the corresponding approximation for pi
p(1) = 16*a(1) - 4*b(1);

% Improve the approximation by increasing the number of terms used
for k = 1:n-1
    a(k + 1) = a(k) + (-1)^k * xA^(2*k+1)/(2*k+1);
    b(k + 1) = b(k) + (-1)^k * xB^(2*k+1)/(2*k+1);
    p(k + 1) = 16*a(k + 1) - 4*b(k + 1);
end
```

Figure 1: Computing ten successive approximations of π using the Machin formula and Taylor series expansions for arctangent.

point values should appear with an appropriate number of digits. Where appropriate, scientific notation should be used, especially if the values are very small or very large. For MATLAB, the formatted print statement (`fprintf`) is ideal for these needs.

As an example, the n approximations we have computed can be tabulated with the following loop:

```
% Tabulate the resulting iterates
fprintf('%3s %18s\n', 'n', 'p(n)')
fprintf('%3s %18s\n', '---', '-----')
for i = 1:n
    fprintf('%3d %18.14f\n', i, p(i))
end
```

The final results are as follows:

```
  n               p(n)
---              -----
  1   3.18326359832636
  2   3.14059702932606
  3   3.14162102932503
  4   3.14159177218218
  5   3.14159268240440
  6   3.14159265261531
  7   3.14159265362355
  8   3.14159265358860
  9   3.14159265358984
 10   3.14159265358979
```