Fractal Math and Graphics

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Talk Overview

- Fractals — Origins and Definition
- Natural Phenomenon
- Motivation for Studying Fractals
- Koch's Snowflake
- Sierpinski's Curves
- Mathematics of Fractals
- Resources and Conclusion

Discovery — Lewis Fry Richardson

- English mathematician, physicist, meteorologist, psychologist, and pacifist, 1881 – 1953
- Decided to research the relation between the probability of two countries going to war and the length of their common border
- He observed the **coastline paradox**: the length of a coastline depends on the level of detail at which you measure it.
- Empirically, the smaller the unit of measure (e.g., ruler vs yardstick), the longer the measured length becomes.

Coastline Paradox

200 KM
Measure

100 KM
Measure

50 KM
Measure

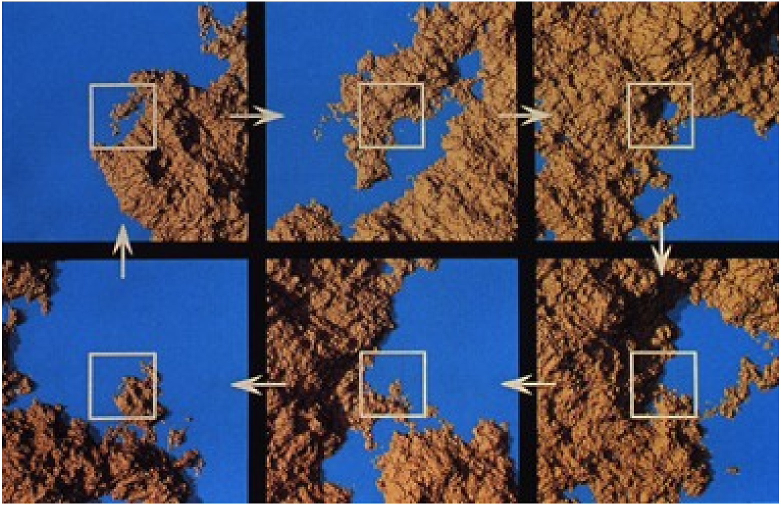
Definition

from Google:

Fractal: a curve or geometric figure, each part of which has the same statistical character as the whole.

Another way of putting it: a fractal is a figure consisting of an identical motif which repeats on an ever–reducing scale.

Coastline Fractal



Chou Romanesco

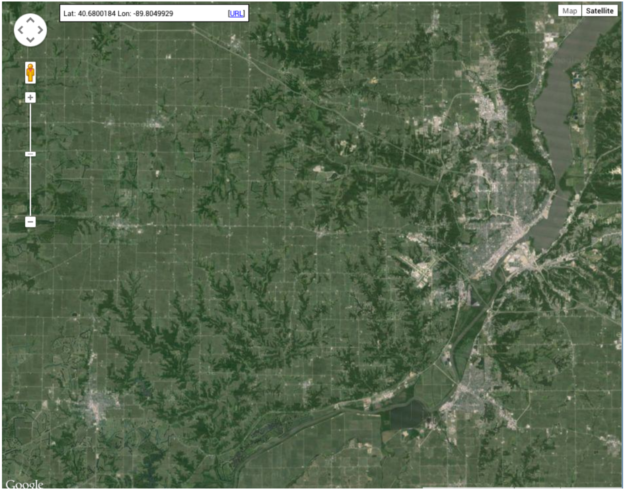
Fractal forms are complex shapes which look more or less the same at a wide variety of scale factors, and are everywhere in nature.

Same Pattern At Macro and Micro Levels

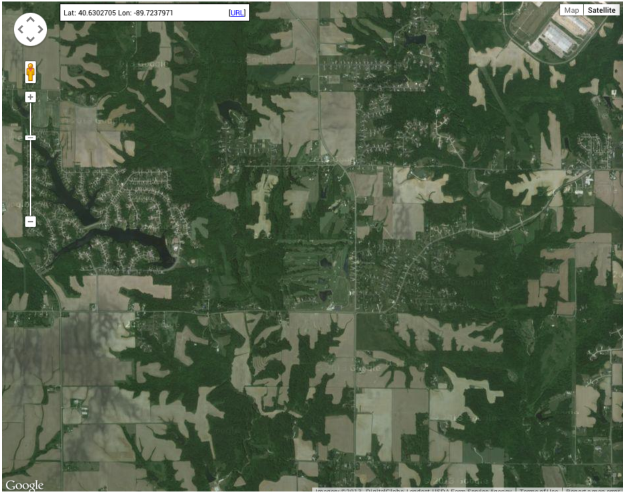
Another Example — Ferns

Frost

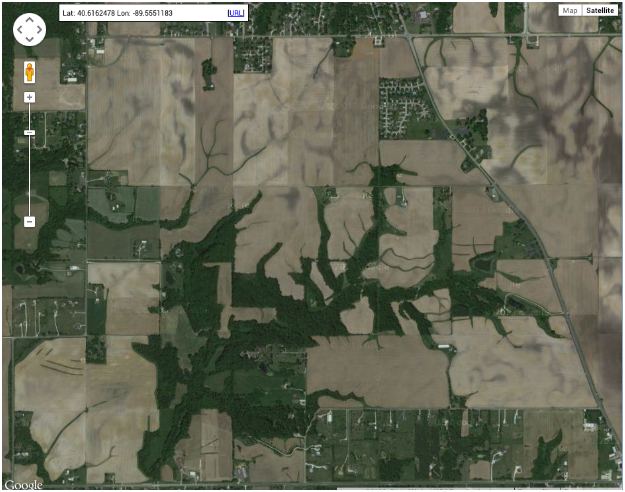
Peoria Via Google Earth



Peoria A Bit Closer. . .



And Closer Yet. . .



Mississippi River Via Google Earth

A Bit Closer...

And Closer Yet. . .

Some Reasons for Studying Fractals

- An interesting way to teach students to look for and understand patterns
- Provide practice with fractions and exponents
- Improve problem solving skills in a colorful environment

More Reasons for Studying Fractals

- Brings visual beauty and art to math and science; allows creativity
- Student motivation — there's a lot about fractals they can understand
- Provides interesting examples of iterative functions and recursive algorithms

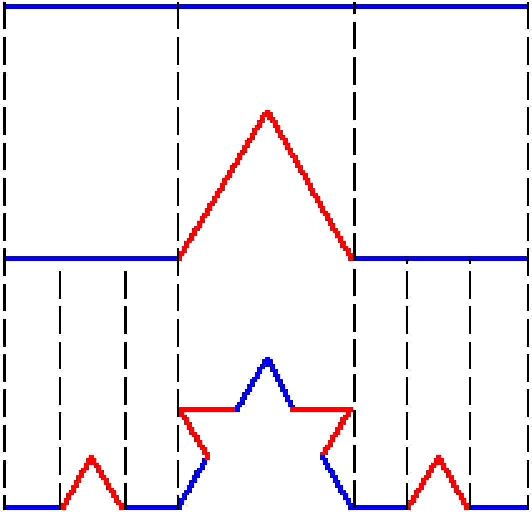
And Yet More Reasons

- Fractals are useful in modeling structures (such as eroded coastlines or snowflakes) in which **similar patterns recur at progressively smaller scales**, and in describing **partly random or chaotic phenomena** such as:

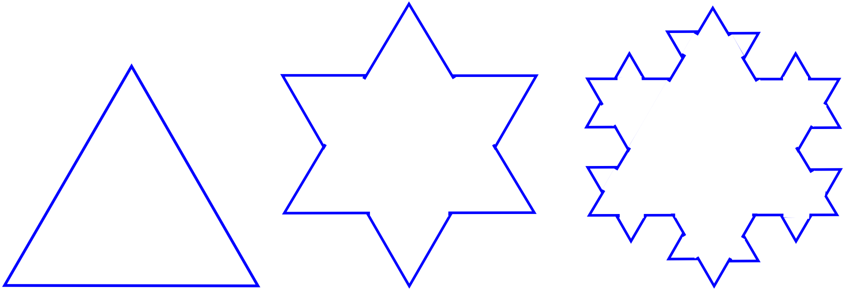
  1. crystal growth,
  2. fluid turbulence, and
  3. galaxy formation.

- They are used to solve real–world problems, for instance in engineering with Fractal Control of Fluid Dynamics

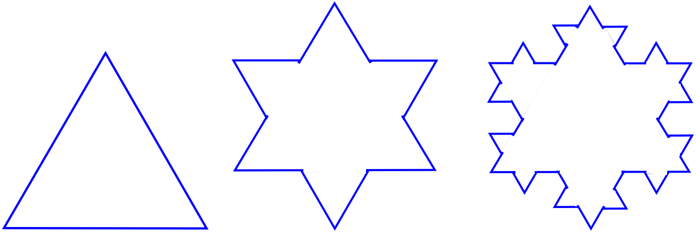
Koch Snowflake Rules



Koch Iterations Over a Triangle



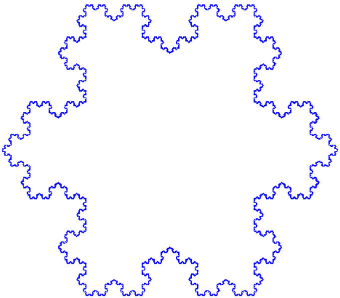
Koch Curve Math

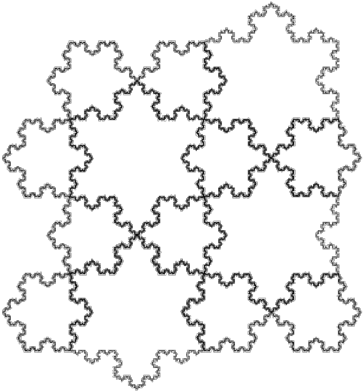


Iteration	1	2	3	4	n
Number of Segments per Side	1	4	16		
Length of Segment	1	$\frac{1}{3}$			
Total Length of Curve	3	4			

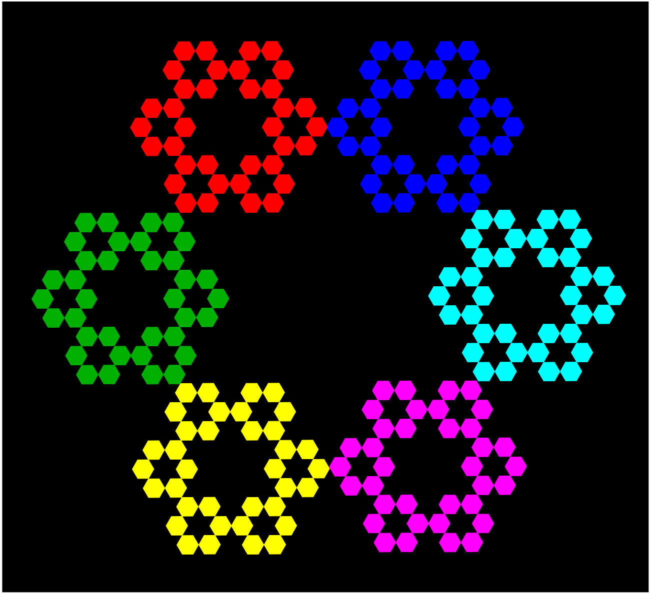
What is the length of the n^{th} iteration?

Further Iterations and Combinations of Koch Snowflakes



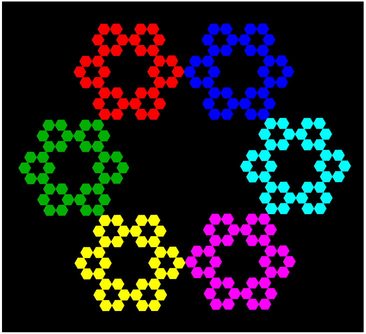


Snowflake by Hexagon

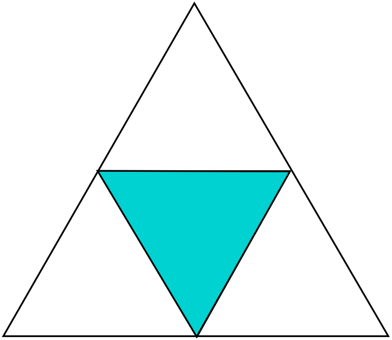


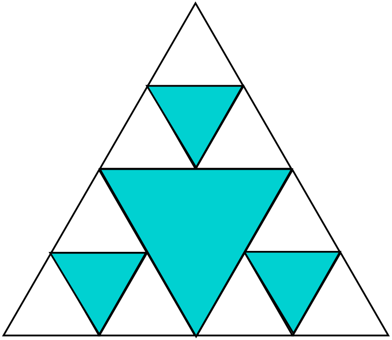
Class Exercise Example

1. The software I use to create graphics took 4 mouse clicks to change the color of each of the small hexagons in the figure at right. Given that I copied the red group of hexagons, how many clicks did it take to change the colors on the copies?
2. The drawing program is a little flakey and tends to have some problems when I "mis-click." If I mis-click on average once out of every 10 mouse clicks, about how many problems occurred?

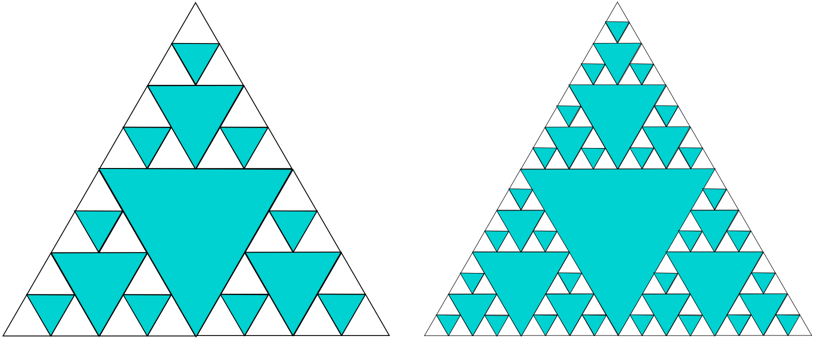


Sierpinski's Triangle — Iterations 1 & 2

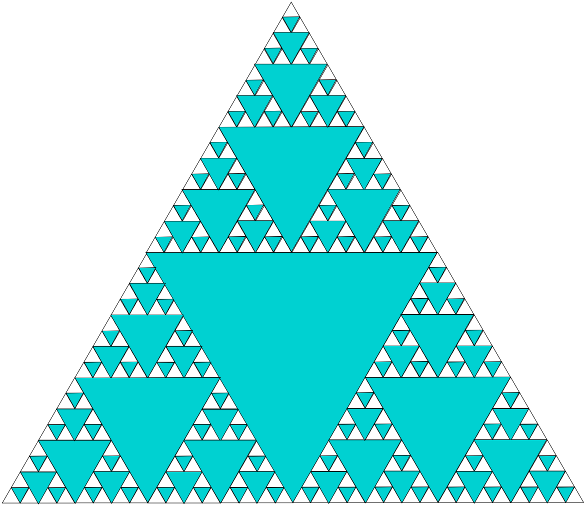




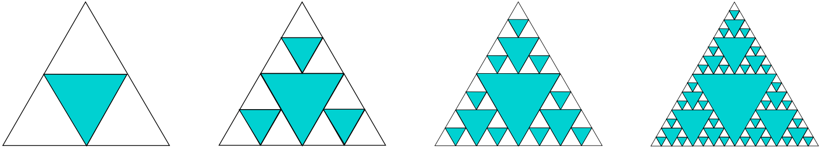
Sierpinski's Triangle — Iterations 3 & 4



Sierpinski's Triangle — Iteration 5



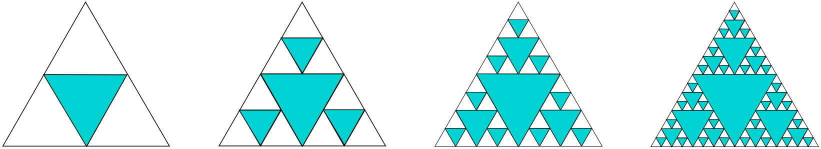
Exploring the Math of Sierpinski's Triangle



Iteration	1	2	3	4	n
Number of Shaded Triangles Added	1	3			
Total Number of Shaded Triangles	1				
Area of Smallest Shaded Triangle	$\frac{1}{4}$				
Total Shaded Area	$\frac{1}{4}$				

Assume the original triangle has area 1.

Possible Student Activities

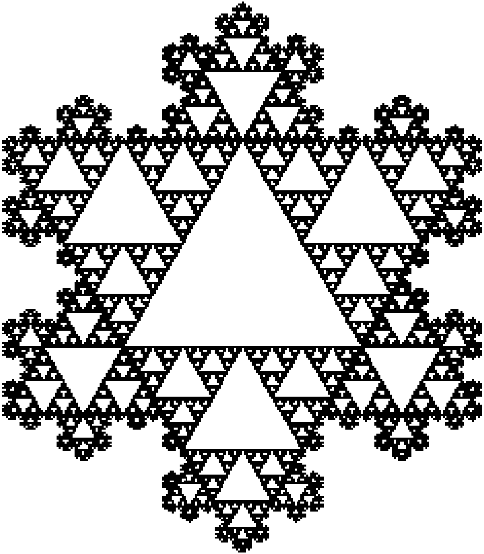


1. Find a pattern for the number of shaded Triangles added at each iteration. Determine a formula for the number of shaded triangles at the n^{th} iteration.

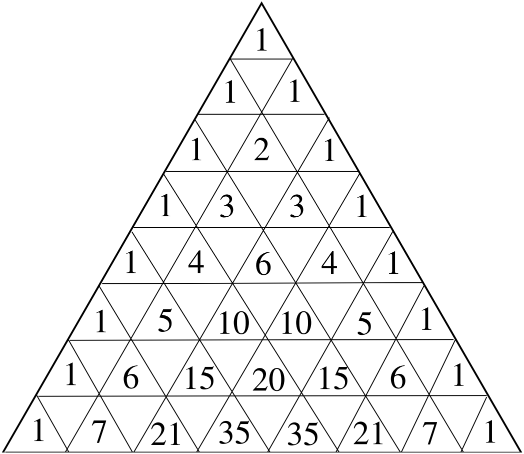
2. Find a pattern for the area of one of the shaded Triangles added at each iteration. Determine a formula for the area of one of the added shaded Triangles at the n^{th} iteration.

3. Find a pattern in the values for the total shaded area. Determine a formula for the total shaded area at the n^{th} iteration.

Combining Koch's Snowflake & Sierpinski's Triangle

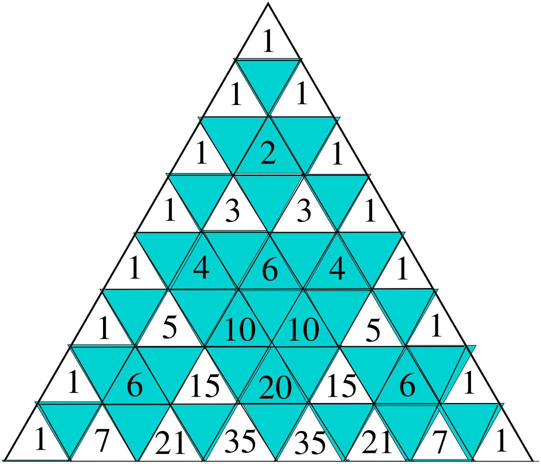


Consider Pascal's Triangle



Binomial Coefficients

Color Empty and Even Triangles

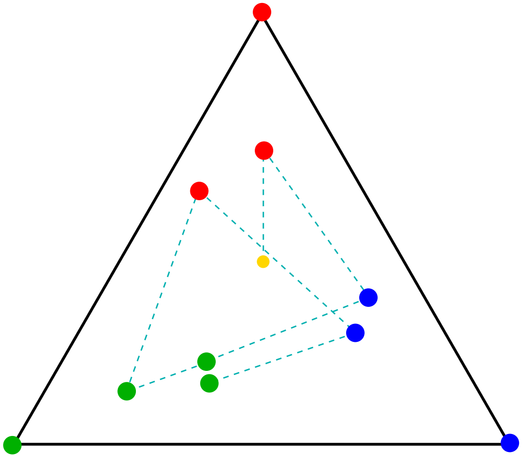


Surprise!

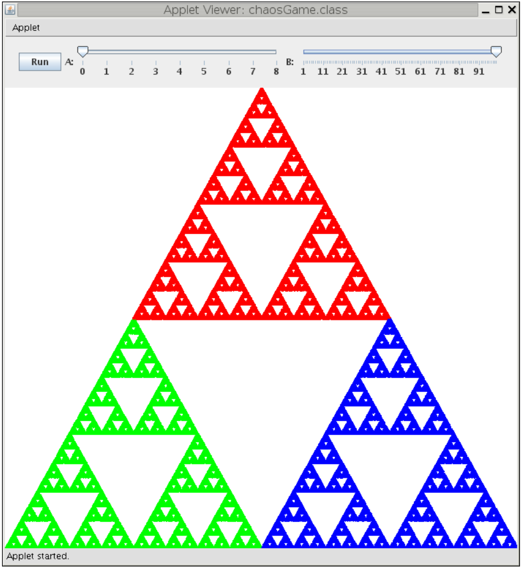
Pascal's Triangle contains Sierpinski's Triangle

How many more rows to get to another complete Sierpinski Triangle?

Seemingly Off Topic. . . The Chaos Game

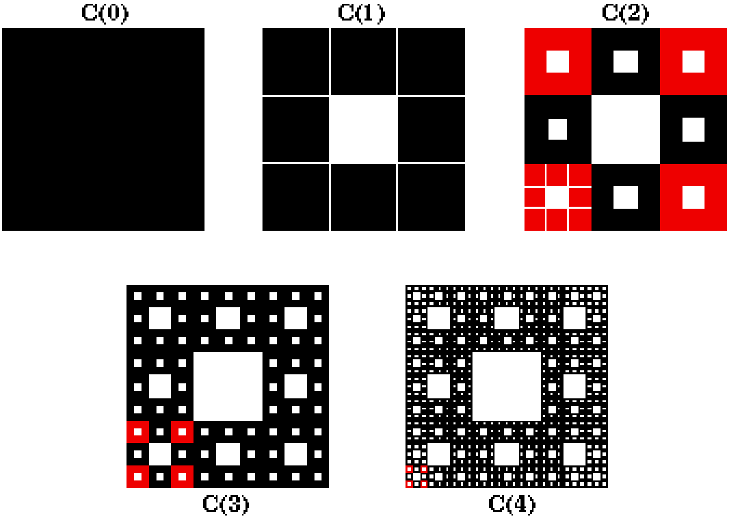


The Result — Sierpinski's Triangle!

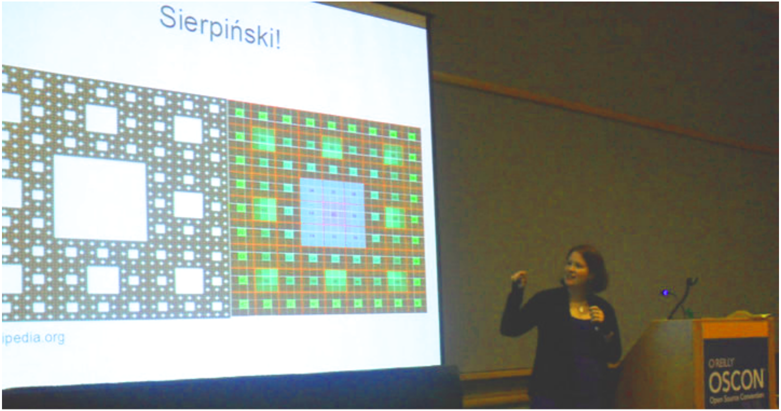


That Triangle Is Everywhere!

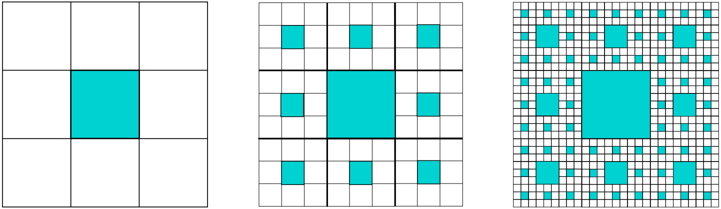
Sierpinski's Carpet



Sierpinski's Quilt



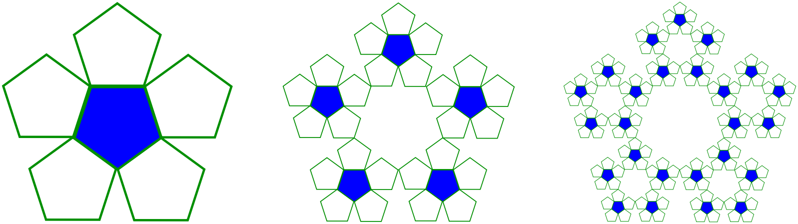
Math & Sierpinski's Carpet



Iteration	1	2	3	n
Number of Shaded Squares Added	1	8		
Total Number of Shaded Squares	1			
Area of Smallest Shaded Square	$\frac{1}{9}$			
Total Shaded Area	$\frac{1}{9}$			

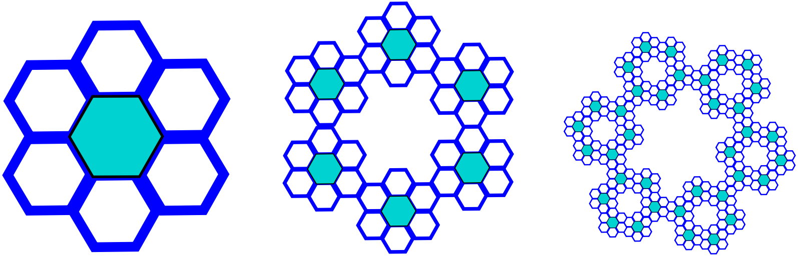
Assume the original square has area 1.

Pentagon Fractal



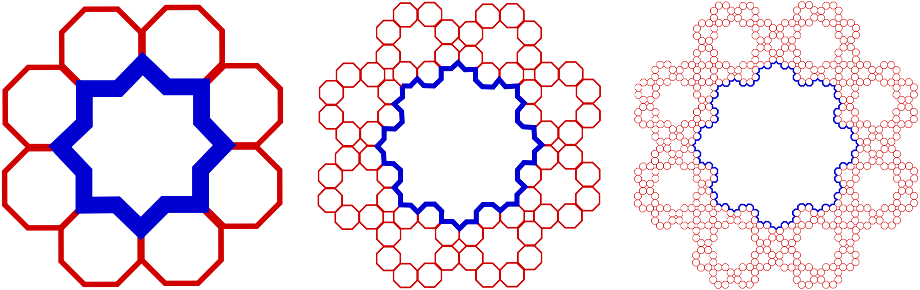
Iteration	1	2	3	n
Total Number of Shaded Pentagons	1	5		
Number of Unshaded Pentagons	5	25		
Total Number of Pentagons	6	30		

Hexagon Fractal



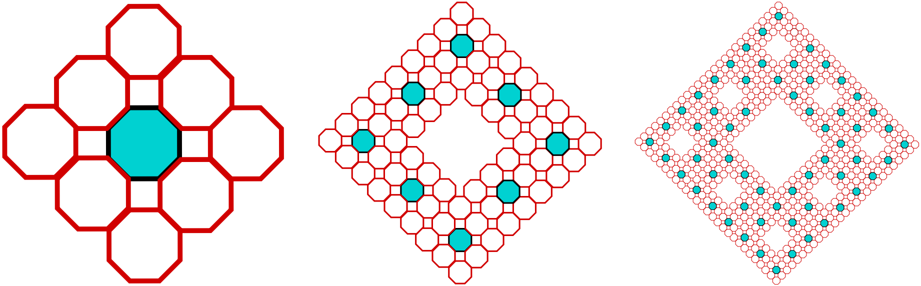
Iteration	1	2	3	n
Total Number of Shaded Hexagons	1	6		
Number of Unshaded Hexagons	5	36		
Total Number of Hexagons	6	42		

Octagon Fractal



Quad–Koch?

Another Octagon Fractal



Octo–Carpet?

Proof Without Words

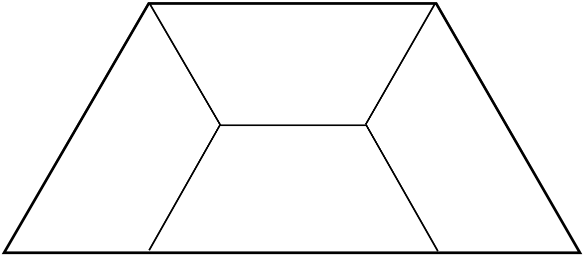
What is the sum of

$$\frac{1}{4} + \left(\frac{1}{4} \textit{of} \frac{1}{4}\right) + \frac{1}{4} \textit{of} \left(\frac{1}{4} \textit{of} \left[\frac{1}{4}\right]\right) + \dots$$

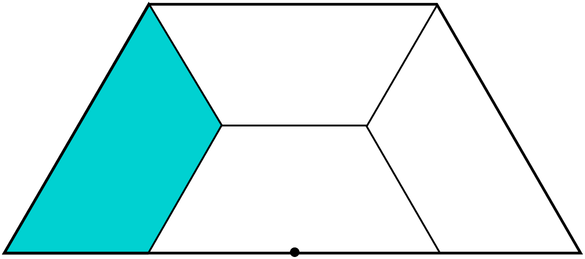
or

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \dots = \frac{1}{4^1} + \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4} + \dots$$

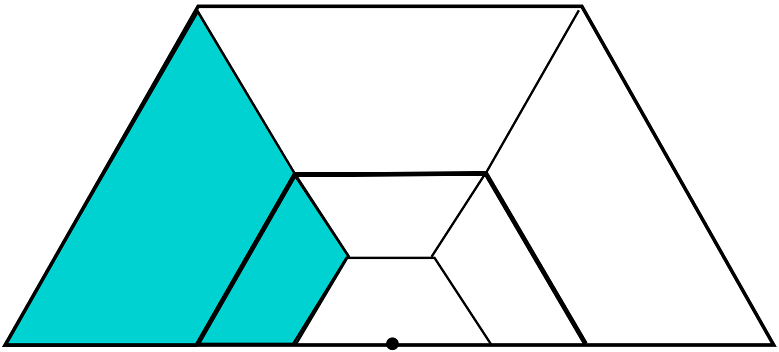
Let us start with one unit, divided into fourths:



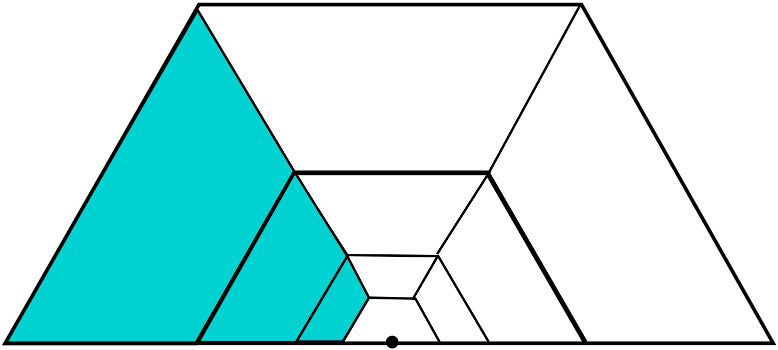
And Keep Shading $\frac{1}{4}$ of Successive Regions



One–fourth of One–fourth



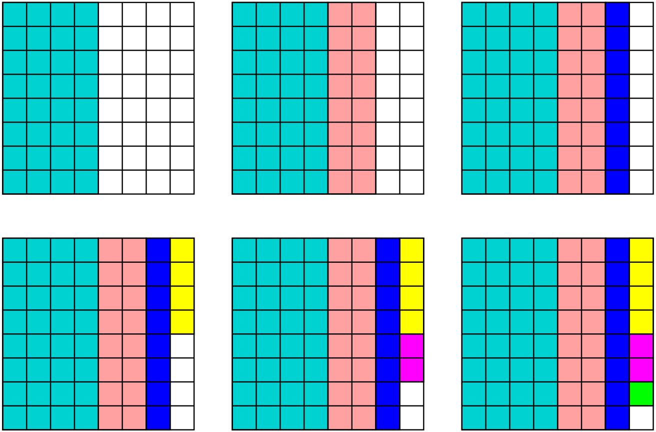
Ah! I See Where This Is Headed!



The pattern continues to repeat at reduced scale.

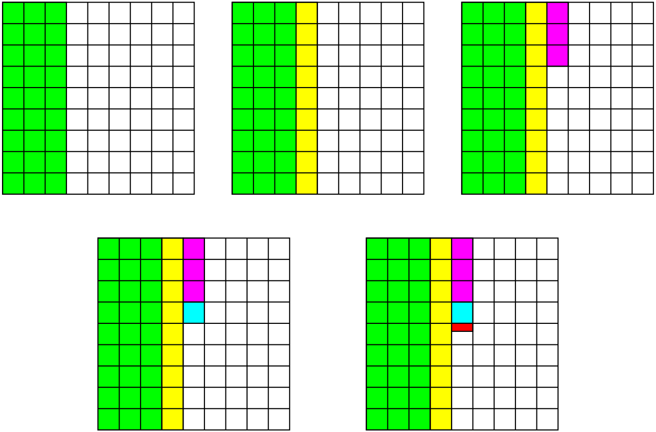
Creeping Forward by Halves

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\ldots$$

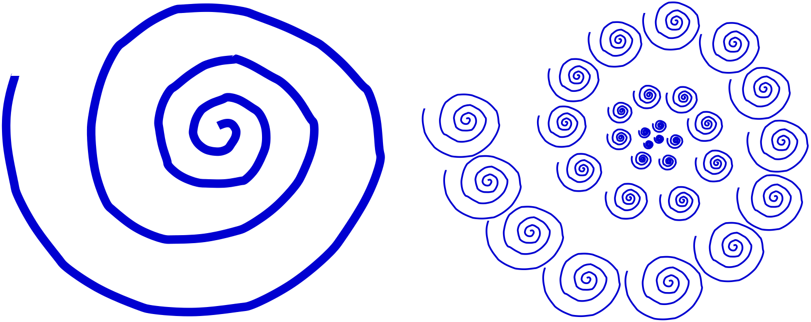


By Thirds...

$$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots=\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\frac{1}{3^4}+\ldots$$

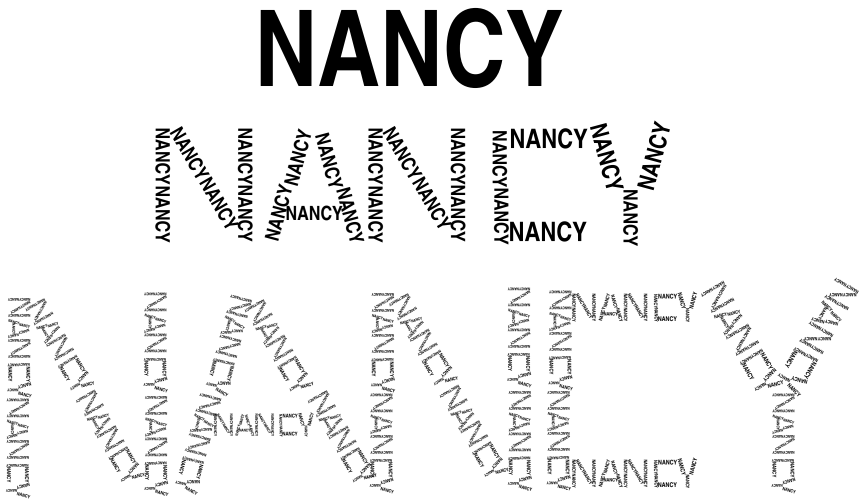


Spiraling Into A Fractal



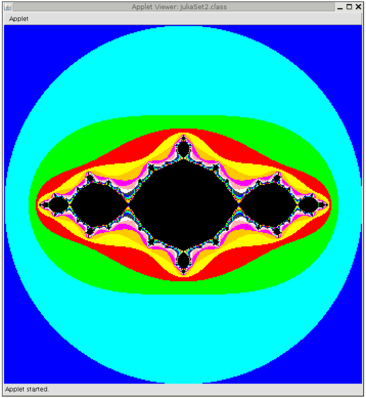
Make a doodle, make a fractal!

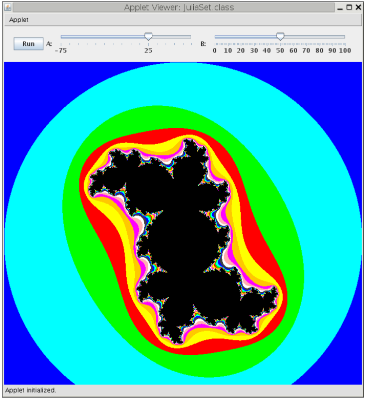
Individualized Fractal Name



How many N's occur in each iteration? How many A's?

The Julia Set





They don't have to be complex. . .

The Julia Map

- To every ordered pair of real values, (a, b), we can associate a **function** of two variables — referred to as the **Julia Map** for (a, b), and denoted by $F_{(a,b)}$

- $F_{(a,b)}$ is described by the formula:

$$F_{(a,b)}(x, y) = (\ x^2 - y^2 + a,\ \ 2xy + b\ ).$$

- Note: when $F_{(a,b)}$ is given a pair of coordinates, it produces another pair:

$$F_{(a,b)}(x, y) = (x', y')$$

Generating Sequences of Points

- For example, if $a = -1$ and $b = 0$:

$$\begin{aligned} F_{(-1,0)}(x, y) &= (x^2 - y^2 + a, 2xy + b) \\ &= (x^2 - y^2 + (-1), 2xy + 0) \\ &= (x^2 - y^2 - 1, 2xy) \end{aligned}$$

- We can start with a point $P_0 = (x_0, y_0)$ and compute the following sequence of coordinate pairs:

$$P_1 = F_{(a,b)}(P_0) = (x_1, y_1),$$
$$P_2 = F_{(a,b)}(P_1) = (x_2, y_2),$$
$$P_3 = F_{(a,b)}(P_2) = (x_3, y_3),$$
$$P_4 = F_{(a,b)}(P_3) = (x_4, y_4), \text{ etc.,}$$

Julia Map Iteration Examples

The beginning sequences for $F_{(-1,0)}$ and three different initial points:

Iterations of $F_{(-1,0)}(x_0, y_0)$			
n	$F^n_{(-1,0)}(0.5, 0.5)$	$F^n_{(-1,0)}(0.5, 0.0)$	$F^n_{(-1,0)}(1.0, 0.0)$
0	(0.5, 0.5)	(0.5, 0.0)	(1.0,0.0)
1	(-1.0, 0.5)	(0.75, 0.0)	(0.0,0.0)
2	(-0.25,-1)	(0.438,0.0)	(-1.0,0.0)
3	(-1.938,0.5)	(-0.809, 0.0)	(0.0,0.0)
4	(2.504,-1.938)	(-0.346, 0.0)	(-1.0,0.0)
5	(1.516,-9.703)	(-0.880, 0.0)	(0.0,0.0)
6	**(-92.844,-29.411)**	**(-0.225, 0.0)**	**(-1.0,0.0)**

Two Types of Behaviors

- Starting at (0.5, 0.5), by the sixth iteration the current point is out in the fourth quadrant of the plane, quite a distance (relatively) from the origin. Successive iterations will move it away even faster.

- On the other hand, starting at each of the other two sample points leads to sequences that stay pretty close to the origin.

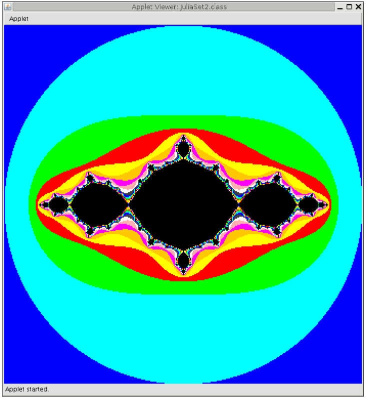
- We observe **two qualitatively different types of behavior**. The sequence of points P_0, P_1, P_2, $P3$, ... either:

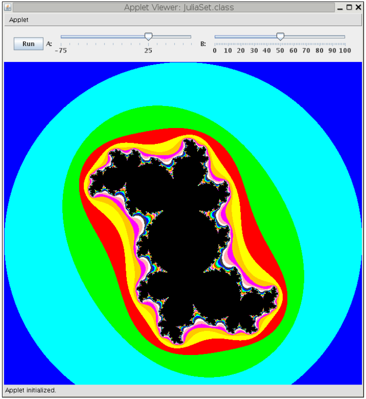
  1. starts to **get farther and farther away** from the origin, or
  2. the sequence **stays pretty close** to the origin

How Colors Are Determined

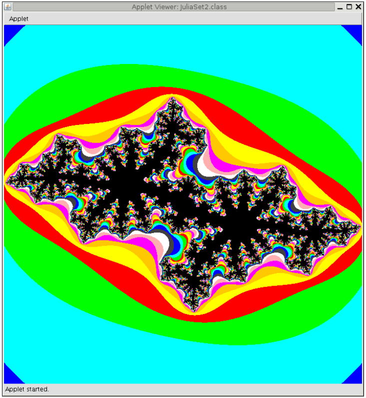
- The **Julia set** for (a, b) is the **collection of all points** in the plane from which you can start and **never get too far away** from the origin by repeated iterations of $F_{(a,b)}$.
- One way to picture these is to color the points in the plane according to **how many iterations** it takes, starting from that point, to get outside a **threshold circle**.
- The points that **don't get out** within a certain, preset number of iterations are the ones that are in the Julia set and they are colored **black**.

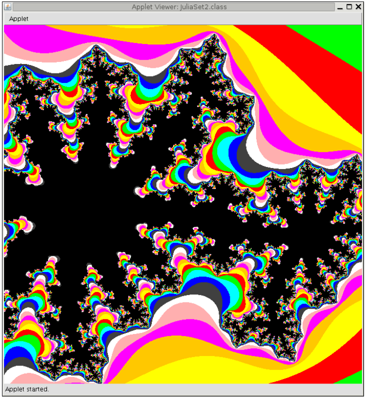
The Results — Using Different Values for a & b



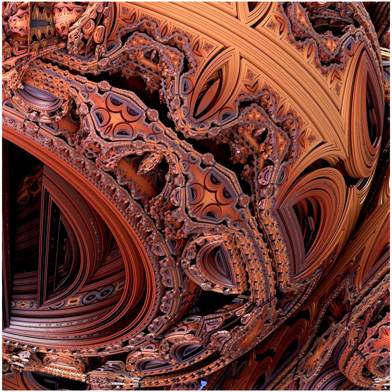


Repeating Patterns

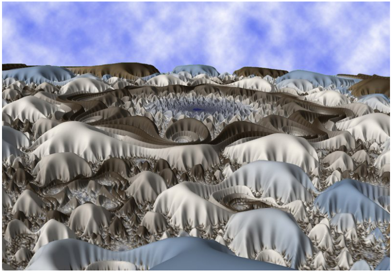




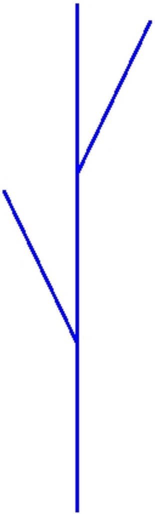
Computer Generated Images

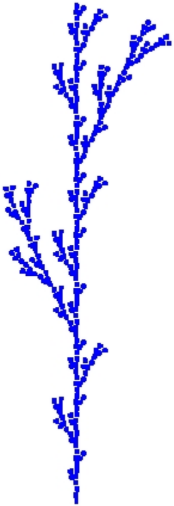


Computer Generated Landscapes

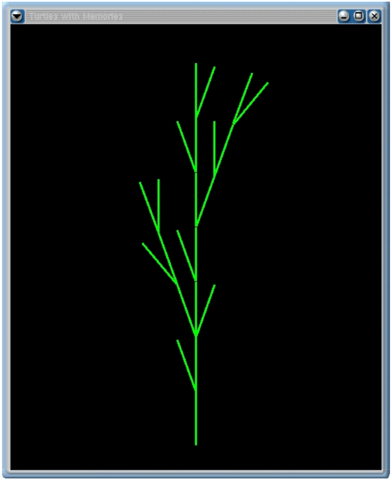


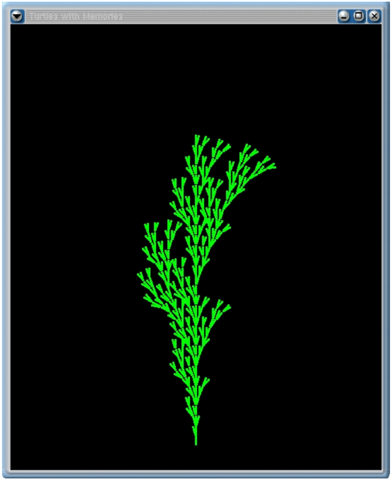
Making Weeds

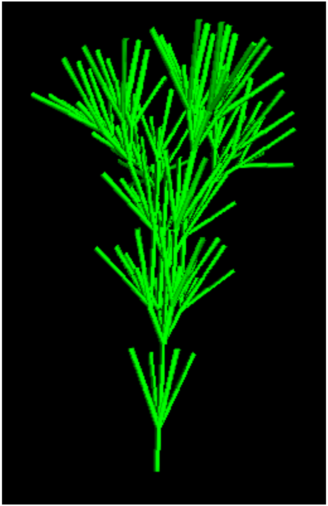




Changing Parameters Changes The Pattern







Online Resources

- **Vi Hart**: vihart.com
  also on YouTube (Vihart) and
  khanacademy.org/math/vi-hart
- **Khan Academy**: khanacademy.org
- Amazing Seattle Fractals: fractalarts.com
- Cynthia Lanius:
  math.rice.edu/~lanius/frac/index.html
- Shodor: www.shodor.org
- Google is your friend!

Thank You!

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