Exercise 2. Rewrite the following polynomials in nested form and evaluate at $x = -1/2$:

(a) $P(x) = 6x^3 - 2x^2 - 3x + 7$

(b) $P(x) = 8x^5 - x^4 - 3x^3 + x^2 - 3x + 1$

Solution. Using synthetic division provides a nice way to see the intermediate computations.

(a) $P(x) = ((6x - 2)x - 3)x + 7 \qquad P(-\frac{1}{2}) = \frac{29}{4}$

$-\frac{1}{2}$	6	-2	-3	7
		-3	$\frac{5}{2}$	$\frac{1}{4}$
	6	-5	$-\frac{1}{2}$	$\frac{29}{4}$

(b) $P(x) = ((((8x - 1)x - 3)x + 1)x - 3)x + 1 \qquad P(-\frac{1}{2}) = \frac{45}{16}$

$-\frac{1}{2}$	8	-1	-3	1	-3	1
		-4	$\frac{5}{2}$	$\frac{1}{4}$	$-\frac{5}{8}$	$\frac{29}{16}$
	8	-5	$-\frac{1}{2}$	$\frac{5}{4}$	$-\frac{29}{8}$	$\frac{45}{16}$

Exercise 3. Evaluate $P(x) = x^6 - 4x^4 + 2x^2 + 1$ at $x = 1/2$ by considering $P(x)$ as a polynomial in x^2 and using nested multiplication.

Solution. Let $u = x^2$. Then $Q(u) = u^3 - 4u^2 + 2u + 1 = ((u - 4)u + 2)u + 1$ provides a way to obtain $P(x)$. As above, synthetic division provides a useful way to visualize the computations described by the nested multiplication:

$\frac{1}{4}$	1	-4	2	1
		$\frac{1}{4}$	$-\frac{15}{16}$	$\frac{17}{64}$
	1	$-\frac{15}{4}$	$\frac{17}{16}$	$\frac{81}{64}$

Thus, $P(\frac{1}{2}) = \frac{81}{64}$.

Exercise 6. Explain how to evaluate the polynomial for a given input x, using as few operations as possible. How many multiplications and how many additions are required?

(a) $P(x) = a_0 + a_5x^5 + a_{10}x^{10} + a_{15}x^{15}$

(b) $P(x) = a_7x^7 + a_{12}x^{12} + a_{17}x^{17} + a_{22}x^{22} + a_{27}x^{27}$

Solution. This is similar to Exercise 3.

(a) Let $u = x^5$. Then $Q(u) = a_0 + a_5u + a_{10}u^2 + a_{15}u^3 = ((a_{15}u + a_{10})u + a_5)u + a_0$ gives the value of $P(x)$. Evaluation of $Q(u)$ takes 3 multiplications and 3 additions. We also need to compute x^5, as follows: $x^5 = (x^2)^2x$, using 3 more multiplications. Altogether, we used 6 multiplications and 3 additions.

(b) Removing a factor of x^7 leaves a polynomial in x^5, which can be rearranged in nested form:

$$P(x) = x^7(((a_{27}x^5 + a_{22})x^5 + a_{17})x^5 + a_{12})x^5 + a_7$$

With 3 multiplications, we can compute $x^5 = (x^2)^2x$. We can get x^7 with one more multiplication (by x^2, which we already know), for a total of 4 multiplications so far. The nested multiplication above takes 4 multiplications and 4 additions, bringing the count to 8 multiplications and 4 additions. Since a multiplication by x^7 is also required, this brings the total to 9 multiplications and 4 additions.