Approximating π with Machin's Formula

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John Machin

- 1680–1751
- English mathematician and astronomer
- Private tutor to Brook Taylor
- Best known for formulas he invented for calculating π

Line drawing from MacTutor History of Mathematics archive

Mathematical underpinnings

Taylor's series for arctangent

$$\arctan x = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$
$$= \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+1}}{2j+1}$$

Machin's formula

$$\pi = 16 \arctan \frac{1}{5} - 4 \arctan \frac{1}{239}$$

Partial sums

$$\arctan x = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+1}}{2j+1}$$

$$a_1 = \frac{x}{1} \qquad j \text{ runs from 0 to 0}$$

$$a_2 = \frac{x}{1} - \frac{x^3}{3} \qquad j \text{ runs from 0 to 1}$$

$$a_3 = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} \qquad j \text{ runs from 0 to 2}$$

$$a_4 = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \qquad j \text{ runs from 0 to 3}$$

$$\dots$$

$$a_{k+1} = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{(-1)^k x^{2k+1}}{2k+1}$$

Computing partial sums with a recurrence

$$a_1 = \frac{x}{1}$$
$$a_2 = a_1 - \frac{x^3}{3}$$
$$a_3 = a_2 + \frac{x^5}{5}$$
$$a_4 = a_3 - \frac{x^7}{7}$$
$$\dots$$

$$a_{k+1} = a_k + \frac{(-1)^k x^{2k+1}}{2k+1}, k \geq 1$$

MATLAB code

```
% Arguments for atan()
xA = 1/5;
xB = 1/239;

% Total number of desired approximations
n = 10;

% atan approximations for xA and xB using just one term
a(1) = xA;
b(1) = xB;

% ...and the corresponding approximation for pi
p(1) = 16*a(1) - 4*b(1);

% Improve the approximation by increasing the number of terms used
for k = 1:n-1
    a(k + 1) = a(k) + (-1)^k * xA^(2*k+1)/(2*k+1);
    b(k + 1) = b(k) + (-1)^k * xB^(2*k+1)/(2*k+1);
    p(k + 1) = 16*a(k + 1) - 4*b(k + 1);
end
```

Some numerical results

n	p(n)
1	3.18326359832636
2	3.14059702932606
3	3.14162102932503
4	3.14159177218218
5	3.14159268240440
6	3.14159265261531
7	3.14159265362355
8	3.14159265358860
9	3.14159265358984
10	3.14159265358979

Summary

- For centuries, mankind has been fascinated with π.
- How can we compute accurate approximations of π?
- We have observed Machin's formula leads to fast convergence.