Permutations

A permutation of $n$ distinct objects is an arrangement or ordering of the $n$ objects.

The permutations of the letters “abc” are:

- abc, acb, bac, bca, cab, and cba.

In general we could only use some of the objects.

A $r$-permutation of $n$ distinct objects is an arrangement using $r$ of the $n$ objects.

The 1-permutations of the letters “abc” are:

- a, b, and c.

The 2-permutations of the letters “abc” are:

- ab, ac, ba, bc, ca, and cb.
How many permutations are there of the letters “abcde”?

Action 1: Pick a letter
Action 2: Pick a letter
Action 3: Pick a letter
Action 4: Pick a letter
Action 5: Pick a letter

Total

5 × 4 × 3 × 2 × 1 = 120 ways
How many permutations are there of the letters “abcde”?

Action 1: Pick a letter 5 ways
Action 2: Pick a letter 4 ways
Action 3: Pick a letter 3 ways
Action 4: Pick a letter 2 ways
Action 5: Pick a letter 1 ways

Total

\[5 \times 4 \times 3 \times 2 \times 1 = 120\] ways
## Number of Permutations

How many permutations are there of the letters “abcde”?

<table>
<thead>
<tr>
<th>Action 1:</th>
<th>Pick a letter</th>
<th>5 ways</th>
</tr>
</thead>
<tbody>
<tr>
<td>Action 2:</td>
<td>Pick a letter</td>
<td>4 ways</td>
</tr>
<tr>
<td>Action 3:</td>
<td>Pick a letter</td>
<td>3 ways</td>
</tr>
<tr>
<td>Action 4:</td>
<td>Pick a letter</td>
<td>2 ways</td>
</tr>
<tr>
<td>Action 5:</td>
<td>Pick a letter</td>
<td>1 ways</td>
</tr>
</tbody>
</table>

**Total** \[ 5 \times 4 \times 3 \times 2 \times 1 = 120 \text{ ways} \]
We will see products of the form $5 \times 4 \times 3 \times 2 \times 1$ frequently.

The product of consecutive integers from 1 to some $n$ is called a **factorial** and is denoted $n!$.

$$n! = n \times (n - 1) \times (n - 2) \times \cdots \times 2 \times 1$$

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

$$0! = 1$$

There are $n!$ permutations of a set of size $n$. 
Number of $r$-Permutations

How many 3-permutations are there of the letters “abcde”?

Action 1: Pick a letter
Action 2: Pick a letter
Action 3: Pick a letter

Total $= 5 \times 4 \times 3 = 60$ ways

Note that this is like a factorial but we stop before reaching 1.
How many 3-permutations are there of the letters “abcde”?

Action 1: Pick a letter 5 ways
Action 2: Pick a letter 4 ways
Action 3: Pick a letter 3 ways

Total

Note that this is like a factorial but we stop before reaching 1.
How many 3-permutations are there of the letters “abcde”?

Action 1: Pick a letter 5 ways
Action 2: Pick a letter 4 ways
Action 3: Pick a letter 3 ways

Total 5 * 4 * 3 = 60 ways

Note that this is like a factorial but we stop before reaching 1.
How many $r$-permutations are there of a set of size $n$?

Action 1: Pick the first item
Action 2: Pick the second item
Action 3: Pick the third item
   ...
Action $r$: Pick the $r^{th}$ item

Total
How many $r$-permutations are there of a set of size $n$?

Action 1: Pick the first item $n$ ways
Action 2: Pick the second item $n - 1$ ways
Action 3: Pick the third item $n - 2$ ways

\[ \vdots \]

Action $r$: Pick the $r^{th}$ item $n - r + 1$ ways

Total

\[ n \times (n-1) \times \cdots \times (n-r+1) \]
Number of $r$-Permutations

How many $r$-permutations are there of a set of size $n$?

Action 1: Pick the first item \( n \) ways
Action 2: Pick the second item \( n - 1 \) ways
Action 3: Pick the third item \( n - 2 \) ways

\[ \vdots \]
Action \( r \): Pick the \( r^{th} \) item \( n - r + 1 \) ways

Total \( n \times (n - 1) \times \cdots \times (n - r + 1) \)
The number of $r$-permutations is denoted by $P(n, r)$.

$$P(n, r) = n \ast (n - 1) \ast (n - 2) \ast \cdots \ast (n - r + 1)$$

$$= \frac{n!}{(n-r)!}$$

$P(3, 1) = 3$

$P(3, 2) = 6$

$P(3, 3) = 6$

$P(n, 1) = n$

$P(n, n - 1) = P(n, n) = n!$
How many permutations of the letters “aabc” are there?

Note that we can not distinguish the 2 a’s from one another.

Suppose we could distinguish the a’s then the total number of permutations would be $P(4, 4) = 4! = 24$.

$$\text{aabc, aacb, abac, abca, acab, acba,}$$
$$\text{aabc, aacb, abac, abca, acab, acba,}$$
$$\text{baac, baca, caab, caba, bcaa, cbaa}$$
$$\text{baac, baca, caab, caba, bcaa, cbaa}$$

However, since we cannot tell the a’s apart we have over counted by a factor of $2! = 2$ (the number of permutations of the a’s).

Thus, there are only $\frac{4!}{2!} = 12$ distinguishable permutations of the letters “aabc”.
How many permutations of the letters “systems” are there?

Note that we can not distinguish the 3 s’s from one another.

If we could distinguish the s’s then the total number of permutations would be $P(7, 7) = 7! = 5040$.

However, since we cannot tell the s’s apart we have over counted by a factor of $3! = 6$ (the number of permutations of the s’s).

Thus, there are only $\frac{7!}{3!} = 840$ distinguishable permutations of the letters “systems”.
In general if we have a total of $n$ objects of which $n_1$ are of one type $n_2$ of another type, . . . , and $n_k$ of another type, then the number of distinguishable permutations is:

$$\frac{n!}{n_1! \times n_2! \times \cdots \times n_k!}$$

Note $n_1 + n_2 + \cdots + n_k = n$. 
Number of Distinguishable Permutations

Suppose a coin is tossed seven times. How many sequences of 4 heads and 3 tails are possible?

This question is asking how many distinguishable permutations are there of “HHHHTTT”.

The are 4 objects of type “head” and 3 objects of type “tail”.

The number of distinguishable permutations is \( \frac{7!}{4! \times 3!} = 35 \).
With permutations order is important.

With combinations we are looking at a subset of the objects.

A \( r \)-combination of \( n \) distinct objects is an unordered selection or “subset” of \( r \) of the objects.

The 2-combinations of the letters “abc” are:
- ab, ac, and bc.

The number of \( r \)-combinations of \( n \) objects is denoted \( C(n, r) \) or \( \binom{n}{r} \).
A \( r \)-permutation can be formed by choosing any \( r \)-combination of the \( n \) objects and then arranging them in some order.

Number of \( r \)-permutations

1. Choose a \( r \)-combination
   - \( C(n, r) \) ways
2. Pick an ordering
   - \( r! \) ways

Total

Thus, \( C(n, r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!} \).
A $r$-permutation can be formed by choosing any $r$-combination of the $n$ objects and then arranging them in some order.

Number of $r$-permutations

Action 1: Choose a $r$-combination $C(n, r)$ ways
Action 2: Pick an ordering $r!$ ways

Total

Thus, $C(n, r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$. 
A $r$-permutation can be formed by choosing any $r$-combination of the $n$ objects and then arranging them in some order.

Number of $r$-permutations

Action 1: Choose a $r$-combination $C(n, r)$ ways
Action 2: Pick an ordering $r!$ ways

Total

$$P(n, r) = \frac{n!}{(n-r)!}$$

Thus, $C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!}$. 

The Number of $r$-Combinations
A \( r \)-permutation can be formed by choosing any \( r \)-combination of the \( n \) objects and then arranging them in some order.

**Number of \( r \)-permutations**

\[
\begin{align*}
\text{Action 1:} & \quad \text{Choose a} \ r\text{-combination} \quad \frac{n!}{r!(n-r)!} \text{ ways} \\
\text{Action 2:} & \quad \text{Pick an ordering} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qua