MAT2345 Discrete Math

Dr. Van Cleave

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General Guidelines

- Syllabus
- Schedule (note exam dates)
- Homework, Worksheets, Quizzes, and possibly Programs & Reports
- Academic Integrity Guidelines — Do Your Own Work
- Course Web Site: `www.eiu.edu/~mathcs`

Course Overview

An introduction to the mathematical foundations needed by computer scientists.

- Logic & proof techniques
- Sets, functions
- Algorithms – developing and analyzing
- Recursion & induction proofs
- Recurrence relations
- If time permits:
  - Boolean algebra, logic gates, circuits
  - Modeling computation

Course Themes

- **Mathematical Reasoning** – proofs, esp by induction
- **Mathematical Analysis** – comparison of algorithms, function growth rates
- **Discrete Structures** – abstract math structures, the relationship between discrete and abstract structures
- **Algorithmic Thinking** – algorithmic paradigms
- **Applications and Modeling** – can we predict behavior?

Student Responsibilities — Week 1

- **Reading**: Textbook, Sections 1.1 – 1.4
- **Assignments**: See Homework Assignments Handout
- **Attendance**: Strongly Encouraged

Week 1 Overview

- 1.1 Propositional Logic
- 1.2 Propositional Equivalences
- 1.3 Predicates and Quantifiers
- 1.4 Nested Quantifiers

Section 1.1 Propositional Logic

- The rules of logic are used to distinguish between **valid** and **invalid** mathematical arguments.
- Logic rules have many applications in computer science. They are used in:
  - the design of computer circuits
  - the construction of computer programs
  - the verification of the correctness of programs
  - as the basis of some Artificial Intelligence programming languages.
  - and many other ways as well

Propositions

- **PROPOSITION**: a statement that is either true or false, but not both.

  Examples (which are `true`?):
  - The zip code for Charleston, IL is 61920.
  - The Jackson Avenue Coffee Shop is located on Jackson Avenue.
  - $1 + 4 = 5$
  - $1 + 3 = 5$
  - The title of our course is Mathemagics.

  Counterexamples:
  - Where am I?
  - Stop!
  - $x + 2y = 4$

Vocabulary

- **Variables** are generally used to represent propositions: $p, q, r, s, \ldots$
- **Tautology**: a proposition which is always **true**.
- **Contradiction**: a proposition which is always **false**.
- **Compound Proposition**: a new proposition formed from existing propositions using **logical operators** (aka **connectives**).
- **Negation**: let p be a proposition. The negation of p is the proposition "**It is not the case that** p," denoted by $\neg p$ or $\sim p$.

Truth Tables

Truth Tables display the relationship between the truth values of propositions.

The truth table for **negation**:

p	$\neg p$
T	F
F	T

When proposition p is true, its negation is false.
When it is false, its negation is true.

The negation of "**Today is Monday**" is "**Today is not Monday**" or "**It is not the case that today is Monday**"

Conjunction

Conjunction: the compound proposition p **and** q, or $p \wedge q$ which is **true** when both p and q are **true** and **false** otherwise.

Let $p =$ **Today is Monday**, and $q =$ **It is raining**.
What is the value of each of the following conjunctions?

$$p \wedge q$$

$$p \wedge \neg q$$

Disjunction

Disjunction: the compound proposition p **or** q, or $p \vee q$ which is **false** when both p and q are **false** and **true** otherwise.

p	q	$p \wedge q$	$p \vee q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

Exclusive Or

Exclusive Or: $p \oplus q$, the proposition that is **true** when exactly **one** of p and q is **true**, and is **false** otherwise.

- "Fries or baked potato come with your meal"
- "Do the dishes or go to your room"

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Implication

Implication: $p \to q$ (IF P THEN Q), the proposition that is **true** *unless* p is **true** and q is **false** (i.e., $T \to F$ is **false**).

p is the **antecedent** or **premise**

q is the **conclusion** or **consequence**

Implication		
p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Implications Related To $p \to q$

Direct Statement	$p \to q$
Converse	$q \to p$
Inverse	$\neg p \to \neg q$
Contrapositive	$\neg q \to \neg p$
Biconditional	$p \leftrightarrow q$ **or** `p iff q` the proposition which is **true** when p and q have the <u>same</u> truth values, and **false** otherwise.

Example

Direct:	If today is Monday, then MAT2345 meets today.
Converse:	If MAT2345 meets today, then today is Monday.
Inverse:	If today is not Monday, then MAT2345 does not meet today.
Contrapositive:	If MAT2345 does not meet today, then today is not Monday.

Converse, Inverse, and Contrapositive

Direct Statement	$p \to q$	If p, then q
Converse	$q \to p$	If q, then p
Inverse	$\sim p \to \sim q$	If not p, then not q
Contrapositive	$\sim q \to \sim p$	If not q, then not p

Let p = "they stay" and q = "we leave"
Direct Statement ($p \to q$, in English):

Converse:

Inverse:

Contrapositive:

Let $p =$ "I surf the web" and $q =$ "I own a PC"

Direct Statement $(p \rightarrow q)$:

Converse:

Inverse:

Contrapositive:

Equivalent Conditionals

		Direct	Converse	Inverse	Contrapositive
		$p \to q$	$q \to p$	$\sim p \to \ \sim q$	$\sim q \to \ \sim p$
p	q	$\sim p \vee q$			
T	T	T	T		
T	F	F	T		
F	T	T	F		
F	F	T	T		

$\Box \to \triangle$ is equivalent to $\sim \Box \vee \triangle$

$$\sim \Box \vee \triangle \equiv \Box \to \triangle$$

$$\Box \vee \triangle \equiv \ \sim \Box \to \triangle$$

Tricky Question

For the expression $p \vee q$, write each of the following in symbols:

Direct Statement:

Converse:

Inverse:

Contrapositive:

Alternate Conditional Forms

Common translations of $p \to q$

If p, then q	p is **sufficient** for q
If p, q	q is **necessary** for p
p implies q	q follows from p
p **only if** q	q if p
q unless $\sim p$	q when p

These translations do not in any way depend upon the truth value of $p \to q$.

Equivalent Expressions

"**If you get home late, then you are grounded**" $\equiv$

You are grounded if you get home late.

Getting home late is sufficient for you to get grounded.

Getting grounded is necessary when you get home late.

Getting home late implies that you are grounded.

Truth Tables for Compound Propositions

The Truth Table of $(p \vee \neg q) \to (p \wedge q)$

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \to (p \wedge q)$
T	T				
T	F				
F	T				
F	F				

System Specifications

Consistent system specifications do not contain **conflicting requirements** that could be used to derive a **contradiction**.

When specifications are not consistent, there is no way to develop a system that satisfies all the specifications.

To determine consistency, first translate the specifications into logical expressions; then determine whether any of the specifications conflict with one another.

Example — Are They Consistent?

System Specifications:

1. Whenever the system software is being upgraded, users cannot access the file system.
2. If users can access the file system then they can save new files.
3. If users cannot save new files, then the system software is not being upgraded.

Translate into Logical Expressions:

$p =$
$q =$
$r =$

$S1 =$
$S2 =$
$S3 =$

Are the Specifications Consistent?

Is there any truth assignment that makes $S1 \land S2 \land S3$ True?

Another Example

1. The system is in multiuser state if and only if it is operating normally.
2. If the system is operating normally, the kernel is functioning.
3. The kernel is not functioning or the system is in interrupt mode.
4. If the system is not in multiuser state, then it is in interrupt mode.
5. The system is not in interrupt mode.

Translate into Logical Expressions

$p =$
$q =$
$r =$
$s =$

$S1 =$
$S2 =$
$S3 =$
$S4 =$
$S5 =$

Are the Specifications Consistent?

What indicates a system is inconsistent?

Logical and Bit Operations

- `bit` $\equiv$ `binary digit`: smallest unit of storage in computer memory, has two possible values — `true` (1) and `false` (0).
- `Boolean Variable`: program unit of storage that can contain one of two values — either `true` or `false`, and can thus be represented by a bit.
- `Bit Operations`: correspond to logical connectives: $\wedge,\ \vee,\ \oplus, \neg$
- `Bit String`: a sequence of zero or more bits. The `length` of the string is the number of bits in it.

  Bitwise `OR`, Bitwise `AND`, and Bitwise `XOR` can be applied to bit strings.

An Exercise

0101 1110	1101 1011	0011 0110	p q
			bitwise OR
			bitwise AND
			bitwise XOR

1.2 Propositional Equivalences

Contingency: a proposition which is neither a **tautology** nor a **contradiction**.

p	$\neg p$	$p \vee \neg p$ tautology	$p \wedge \neg p$ contradiction
T	F	T	F
F	T	T	F

Logical Equivalence

Logically Equivalent: two compound propositions which always have the same truth value (given the same truth assignments to any Boolean Variables).

p	q	$p \wedge q$	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	T	F	F	F	F
T	F	F	T	F	F	T	F
F	T	F	T	F	T	F	F
F	F	F	F	T	T	T	T

Thus, $\neg(p \vee q) \equiv \neg p \wedge \neg q$

On Worksheet Provided

Using the Truth Table provided, show:

$p \wedge q$ is logically equivalent to $\neg[p \rightarrow (\neg q)]$

$p \vee q$ is logically equivalent to $(\neg p) \rightarrow q$

$p \vee (q \wedge r)$ is logically equivalent to $(p \vee q) \wedge (p \vee r)$

Write as a proposition:

`If I go to Harry's or go to the country, I will not go shopping.`

Begin by breaking the compound into separate propositions:

- $\mathtt{H} =$
- $\mathtt{C} =$
- $\mathtt{S} =$

Then write as a compound proposition using $\mathtt{H}$, $\mathtt{C}$, and $\mathtt{S}$:

Name that Term!

What is a proposition which

1. is always `true`? ______________________________
2. is always `false`? ______________________________
3. is neither `1.` nor `2.`? ______________________________

- Two propositions, p and q, are **logically equivalent** if $p \leftrightarrow q$ is a tautology.

- We write $p \Leftrightarrow q$

- Example: $(p \rightarrow q) \wedge (q \rightarrow p) \Leftrightarrow p \leftrightarrow q$

To show a proposition is not a tautology, you may use an *abbreviated* truth table and

- try to find a *counter example* to *disprove* the assertion
- search for a case where the proposition is false

Proving Logical Equivalence

Prove these expressions are logically equivalent:

$$(p \to q) \wedge (q \to p) \quad \Leftrightarrow \quad p \leftrightarrow q$$

When would they **not** be equivalent?

Case 1. left side false, right side true...

Subcase a. $p \to q$ is false

Subcase b. $q \to p$ is false

Case 2. left side true, right side false...

Subcase a. $p = \mathtt{T}$, $q = \mathtt{F}$

Subcase b. $p = \mathtt{F}$, $q = \mathtt{T}$

There are no more possibilities, so the two propositions must be logically equivalent.

Note Tables 6, 7, & 8 in section 1.2 — these are **important** for simplifying propositions and proving logical equivalences.

The Porsche & The Tiger

A prisoner must make a choice between two doors: behind one is a beautiful red Porsche, and behind the other is a hungry tiger. Each door has a sign posted on it, but only one sign is true.

Door #1. `In this room there is a Porsche and in the other room there is a tiger.`

Door #2. `In one of these rooms there is a Porsche and in one of these rooms there is a tiger.`

With this information, the prisoner is able to choose the correct door... Which one is it?

In Review

$\sim p$	**negation** of p	truth value is opposite of p
$p \wedge q$	**conjunction**	true only when both p and q are true
$p \vee q$	**disjunction**	false only when both p and q are false
$p \rightarrow q$	**conditional**	false only when p is true and q is false
$p \leftrightarrow q$	**biconditional**	true only when p and q have the *same* truth value.

1.3 Predicates

- **Propositional Function** or **Predicate**: a generalization of a proposition which contains one or more variables.
- Predicates become propositions once every variable is **bound** by:
  - **Assigning it a value** from the **Universe of Discourse, U**, or
  - **Quantifying it**

Example 1

- Let $U = Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the integers, and let $P(x):\ x > 0$ be a predicate.
  It has no truth value until the variable x is bound.

- Examples of propositions where x is assigned a value:
  - $P(-3)$
  - $P(0)$
  - $P(3)$

  What is the truth value of each?

More Examples

- $P(y) \lor \neg P(0)$ is **not** a proposition.
  The variable y has not been bound.

- Let R be the 3–variable predicate:
  $R(x, y, z) : x + y = z$

  What is the truth of:

  - $R(2, -1, 5)$
  - $R(3, 4, 7)$
  - $R(x, 3, z)$

Quantifiers

- **Quantifiers** are used to assert that a predicate
  - is true for every value in the Universe of Discourse,
  - is true for some value(s) in the Universe of Discourse, or
  - is true for one and only one value in the Universe of Discourse
- The **Universal quantification of $P(x)$** is the proposition that $P(x)$ is true **for every x** in the Universe of Discourse
- Universal quantification is written as: $\forall x \;\; P(x)$
- For example, let $U = \{1, 2, 3\}$. Then
$$\forall\ x \;\; P(x) \;\Leftrightarrow\; P(1) \wedge P(2) \wedge P(3).$$

- The statement **Every math student studies hard.** can be expressed as:

$$\forall\ x\ P(x)$$

if we let $P(x)$ denote the statement x **studies hard**, and let $U = \{all\ math\ students\}$.

We can also write this statement as:

$$\forall\ x\ (S(x) \rightarrow P(x))$$

if we let $S(x)$ denote the statement x **is a math student**, and $P(x)$ and U are as before.

Existential Quantification

- **Existential quantification** asserts a proposition is true if and only if it is true for at least one value in the universe of discourse.
- The **Existential quantification of $P(x)$** is the proposition that $P(x)$ is true **for some x** in the Universe of Discourse
- Existential quantification is written as: $\exists x \;\; P(x)$
- For example, let $U = \{1, 2, 3\}$. Then
  $$\exists\, x \;\; P(x) \;\Leftrightarrow\; P(1) \vee P(2) \vee P(3).$$

Unique Existential Quantification

- **Unique Existential Quantification** asserts a proposition is true for **one and only one** $x \in U$, and is written $\exists\, !\, x \;\; P(x)$
- **Remember**: a predicate is **not** a proposition until **all** variables have been bound either by quantification or assignment of a value.

Equivalences Involving Negation

$$\neg \forall x \;\; P(x) \;\Leftrightarrow\; \exists x \;\; \neg P(x)$$

"*$P(x)$ is not true for all x*" **is logically equivalent to** "*there is some x for which $P(x)$ is not true*"

$$\neg \exists x \;\; P(x) \;\Leftrightarrow\; \forall x \;\; \neg P(x)$$

"*There is no x for which $P(x)$ is true*" **is logically equivalent to** "*$P(x)$ is not true for every x*"

- Distributing a negation operator across a quantifier changes a universal to an existential, and vice versa
- If there are multiple quantifiers, they are read from left to right

Nested Quantification Examples

Multiple quantifiers are read from left to right.

Let $U = \mathbb{R}$, the real numbers. Then consider $P(x, y): \; xy = 0$

Which of the following are TRUE?

- $\forall x \;\; \forall y \;\; P(x, y)$
- $\forall x \;\; \exists y \;\; P(x, y)$
- $\exists x \;\; \forall y \;\; P(x, y)$
- $\exists x \;\; \exists y \;\; P(x, y)$

Suppose $P(x, y) : \frac{x}{y} = 1 \ldots$ now which are TRUE?

Conversions

Let $U = \{1, 2, 3\}$. Find an expression equivalent to:

$$\forall\, x \;\; \exists\, y \;\; P(x, y)$$

where the variables are bound by substitution instead of quantification.

We can expand from the inside out, or the outside in...

Outside in, we get:

$$\exists\, y \;\; P(1, y) \;\; \wedge \;\; \exists\, y \;\; P(2, y) \;\; \wedge \;\; \exists\, y \;\; P(3, y) \quad \Leftrightarrow$$

$$\begin{aligned} &[\; P(1,1) \vee P(1,2) \vee P(1,3) \;] \;\wedge \\ &[\; P(2,1) \vee P(2,2) \vee P(2,3) \;] \;\wedge \\ &[\; P(3,1) \vee P(3,2) \vee P(3,3) \;] \end{aligned}$$

Translating English To Symbols, I

Let $U = \{$ all EIU students $\}$, and

$F(x)$: x speaks French fluently

$J(x)$: x knows Java

1. Someone can speak French and knows Java

$$\exists\, x \ \ (F(x) \ \wedge \ J(x))$$

2. Someone speaks French, but doesn't know Java
3. Everyone can either speak French or knows Java
4. No one speaks French or knows Java
5. If a student knows Java, they can speak French

Translating English to Symbols, II

Let $U = \{$ fleegles, snurds, thingamabobs $\}$, and

$F(x)$: x is a fleegle

$S(x)$: x is a snurd

$T(x)$: x is a thingamabob

1. Everything is a fleegle

$$\forall x \ \ F(x) \quad \Leftrightarrow \quad \neg\exists x \ \ \neg F(x)$$

2. Nothing is a snurd
3. All fleegles are snurds
4. Some fleegles are thingamabobs
5. No snurd is a thingamabob
6. If any fleegle is a snurd then it's also a thingamabob

Commutivity & Distribution of Quantifiers

- When all quantifiers are the same, they may be interchanged:

  CORRECT : $\forall x \forall y P(x, y) \Leftrightarrow \forall y \forall x P(x, y)$

  WRONG : $\forall x \exists y P(x, y) \Leftrightarrow \exists y \forall x P(x, y)$

- A quantifier may be distributed over $\wedge$ and $\vee$, but **not** over an implication:

  CORRECT : $\forall x [P(x) \wedge Q(x)] \Leftrightarrow \forall x P(x) \wedge \forall x Q(x)$

  WRONG : $\forall x [P(x) \rightarrow Q(x)] \Leftrightarrow [\forall x P(x) \rightarrow \forall x Q(x)]$