Mat2345
Week 2

Chap 1.5, 1.6

Fall 2013

Student Responsibilities — Week 2

- **Reading**: Textbook, Sections 1.5 – 1.6
- **Assignments**: as given in the Homework Assignment list (handout) — Secs. 1.5 & 1.6
- **Attendance**: Dryly Encouraged

Week 2 Overview

- Finish up 1.1–1.4
- 1.5 Rules of Inference
- 1.6 Introduction to Proofs

Negating Quantifiers

Care must be taken when negating statements with quantifiers.

Negations of Quantified Statements	
Statement	**Negation**
All do	**Some do not** (Equivalently: **Not all do**)
Some do	**None do** (Equivalently: **All do not**)

Practice with Negation

What is the negation of each statement?

1. **Some** people wear glasses.
2. **Some** people **do not** wear glasses.
3. **Nobody** wears glasses.
4. **Everybody** wears glasses.
5. **Not everybody** wears glasses.

Some Notes of Interest

- DeMorgan's Laws:

$$\neg(p \wedge q) \equiv \neg p \vee \neg q \qquad \neg(p \vee q) \equiv \neg p \wedge \neg q$$

- $p \to q$ is **false** only when p is **true** and q is **false**
- $p \to q \equiv \neg p \vee q$
- The negation of $p \to q$ is $p \wedge \neg q$

Which Are Equivalent?

				Direct	Inverse	Converse	Contrapositive
p	q	$\neg p$	$\neg q$	$p \to q$	$\neg p \to \neg q$	$q \to p$	$\neg q \to \neg p$
T	T	F	F				
T	F	F	T				
F	T	T	F				
F	F	T	T				

1.5 Rules of Inference
Theorems, Lemmas, & Corollaries

- A **theorem** is a **valid** logical assertion which can be proved using:
  - other theorems
  - **axioms** : statements given to be true
  - **Rules of Inference** : logic rules which allow the deduction of conclusions from premises.
- A **lemma** is a **pre–theorem** or result which is needed to prove a theorem.
- A **corollary** is a **post–theorem** or result which follows directly from a theorem.

Mathematical Proofs

- Proofs in mathematics are valid arguments that establish the truth of mathematical statements.

- **Argument** : a sequence of statements that ends with a conclusion.

- **Valid** : the conclusion or final statement of the argument must follow from the truth of the preceding statements, or **premises**, of the argument.

- An argument is valid if and only if it is impossible for all the premises to be true and the conclusion to be false.

```
If it rains, then the squirrels will hide
It is raining.
-----------------------------------------
The squirrels are hiding.

p = it rains / is raining
q = the squirrels hide / are hiding
```

Premise 1: $p \to q$ **Premise 2**: p **Conclusion**: q

Associated Implication: $((p \to q) \wedge p) \to q$

p	q	$((p \to q) \wedge p) \to q$
T	T	
T	F	
F	T	
F	F	

Are the squirrels hiding?

```
If you come home late, then you are grounded.
You come home late.
---------------------------------------------
You are grounded.
p =
q =
```

Premise 1:

Premise 2:

Conclusion:

Associated Implication:

p	q	
T	T	
T	F	
F	T	
F	F	

Are you grounded?

Modus Ponens — The Law of Detachment

Both of the prior examples use a pattern for argument called **modus ponens**, or **The Law of Detachment**.

$$\begin{array}{c} p \rightarrow q \\ p \\ \hline q \end{array}$$

or

$$((p \rightarrow q) \wedge p) \rightarrow q$$

Notice that **all** such arguments lead to **tautologies**, and therefore are **valid**.

```
If a knee is skinned, then it will bleed.
This knee is skinned.
-----------------------------------------
It will bleed.
p =
q =
```

Premise 1:

Premise 2:

Conclusion:

Associated Implication:

p	q	
T	T	
T	F	
F	T	
F	F	

(**Modus Ponens**) – *Did the knee bleed?*

Modus Tollens — Example


```
If Frank sells his quota, he'll get a bonus.
Frank doesn't get a bonus.
-------------------------------------
Frank didn't sell his quota.
p =
q =
```

Premise 1: $p \rightarrow q$ **Premise 2**: $\sim q$ **Conclusion**: $\sim p$

Thus, the argument converts to: $((p \rightarrow q) \wedge \sim q) \rightarrow \sim p$

p	q	$((p \rightarrow q) \wedge \sim q) \rightarrow \sim p$
T	T	
T	F	
F	T	
F	F	

Did Frank sell his quota or not?

Modus Tollens

An argument of the form:

$$\begin{array}{c} p \rightarrow q \\ \sim q \\ \hline \sim p \end{array}$$

or

$$((p \rightarrow q) \wedge \sim q) \rightarrow \sim p$$

is called **Modus Tollens**, and represents a **valid** argument.

```
If the bananas are ripe, I'll make banana bread.
I don't make banana bread.
-------------------------------------
The bananas weren't ripe.
p =
q =
```

Premise 1: $p \to q$ **Premise 2**: $\sim q$ **Conclusion**: $\sim p$

Thus, the argument converts to: $((p \to q) \wedge \sim q) \to \sim p$

p	q	$((p \to q) \wedge \sim q) \to \sim p$
T	T	
T	F	
F	T	
F	F	

Were the bananas ripe or not?

Other Famous Rules of Inference

p $\therefore p \vee q$	Addition
$p \wedge q$ $\therefore p$	Simplification
$p \rightarrow q$ $q \rightarrow r$ $\therefore p \rightarrow r$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	Disjunctive syllogism
p q $\therefore p \wedge q$	Conjunction
$(p \rightarrow q) \wedge (r \rightarrow s)$ $p \vee r$ $\therefore q \vee s$	Constructive dilemma

Rules of Inference for Quantifiers

$\forall x P(x)$ $\therefore P(c)$	Universal Instantiation (UI)
$P(c)$ (for arbitrary c) $\therefore \forall x P(x)$	Universal Generalization (UG)
$P(c)$ (for some c) $\therefore \exists x P(x)$	Existential Generalization
$\exists x P(x)$ $\therefore P(c)$ (for some c)	Existential Instantiation

- In Universal Generalization, x must be arbitrary.
- In Universal Instantiation, c need not be arbitrary but often is assumed to be.
- In Existential Instantiation, c must be an element of the universe which makes $P(x)$ true.

Proof Example


```
Every human experiences challenges.
Kim Smith is a human.
-------------------------------------
Kim Smith experiences challenges.

H(x) = x is a human
C(x) = x experiences challenges
k = Kim Smith, a member of the universe
```

Predicate 1: $\forall x[H(x) \to C(x)]$ **Predicate 2**: $H(k)$ **Conclusion**: $C(k)$

The proof:

(1)	$\forall x[H(x) \to C(x)]$	Hypothesis (1)
(2)	$H(k) \to C(k)$	step (1) and UI
(3)	$H(k)$	Hypothesis 2
(4)	C(k)	steps 2 & 3, and Modus Ponens

Q.E.D.

Fallacies

Fallacies are incorrect inferences.

An argument of the form:

$$\begin{array}{c} p \ \rightarrow q \\ \sim p \\ \text{------} \\ \sim q \end{array}$$

or

$$((p \ \rightarrow \ q) \wedge \ \sim p) \ \rightarrow \ \sim q$$

is called the **Fallacy of the Inverse** or **Fallacy of Denying the Antecedent**, and represents an **invalid** argument.

Fallacy of the Inverse — Example


```
If it rains, I'll get wet.
It doesn't rain.
-------------------------------------
I don't get wet.
p =
q =
```

Premise 1: $p \to q$ **Premise 2**: $\sim p$ **Conclusion**: $\sim q$

Thus, the argument converts to: $((p \to q) \ \wedge \ \sim p) \ \to \ \sim q$

p	q	$((p \to q) \ \wedge \ \sim p) \ \to \ \sim q$
T	T	
T	F	
F	T	
F	F	

Did I get wet?

Did the Butler Do It?

`If the butler is nervous, he did it.`
`The butler is really mellow. (i.e., not nervous)`
`Therefore, the butler didn't do it.`

Translate into symbols:

Another Type of (Invalid) Argument


```
If it rains, then the squirrels hide.
The squirrels are hiding.
-------------------------------------
It is raining.
p = it rains / is raining
q = the squirrels hide / are hiding
```

Premise 1: $p \to q$ **Premise 2**: q **Conclusion**: p

Thus, the argument converts to: $((p \to q) \wedge q) \to p$

p	q	$((p \to q) \wedge q) \to p$
T	T	
T	F	
F	T	
F	F	

(**Fallacy of the Converse**) — *Is it raining?*

Fallacy of the Converse

An argument of the form:

$$\begin{array}{c} p \;\to q \\ q \\ \text{------} \\ p \end{array}$$

or

$$((p \;\to\; q) \wedge\; q) \;\to\; p$$

is sometimes called the **Fallacy of the Converse** or **Fallacy of Affirming the Consequent**, and represents an **invalid** argument.

Begging the Question aka Circular Reasoning

Circular Reasoning occurs when the truth of the statement being proved (or something equivalent) is used in the proof itself.

For example:

Conjecture: *if x^2 is even then x is even.*

Proof:
If x^2 is even, then $x^2 = 2k$ for some k. Then $x = 2m$ for some m. Hence, x must be even.

Synopsis of Some Argument Forms

VALID			
Modus Ponens	**Modus Tollens**	**Disjunctive Syllogism**	**Hypothetical Syllogism**
$p \to q$ p	$p \to q$ $\sim q$	$p \vee q$ $\sim p$	$p \to q$ $q \to r$
q	$\sim p$	q	$p \to r$

INVALID	
Fallacy of the Converse	**Fallacy of the Inverse**
$p \to q$ q	$p \to q$ $\sim p$
p	$\sim q$

Valid or Invalid?


```
Either you get home by midnight, or you're grounded.
You aren't grounded.
-------------------------------------
You got home by midnight.

p =

q =
```

Premise 1: $p \vee q$ **Premise 2**: $\sim q$ **Conclusion**: p

Thus, the argument converts to: $((p \vee q) \ \wedge \ \sim q) \ \rightarrow \ p$

Did you get home by midnight?

Argument type:

Valid or Invalid?

```
If you're good, you'll be rewarded.
You aren't good.
-------------------------------------
You aren't rewarded.

p =

q =
```

Premise 1: $p \to q$ **Premise 2**: $\sim p$ **Conclusion**: $\sim q$

Thus, the argument converts to: $((p \to q) \wedge \sim p) \to \sim q$

Are you rewarded?

Argument type:

Valid or Invalid?

```
If you're kind to people, you'll be well liked.
If you're well liked, you'll get ahead in life.
--------------------------------------------------
If you're kind to people, you'll get ahead in life.
```

```
p = you're kind to people
q = you're well liked
r = you get ahead in life
```

Premise 1: $p \to q$ **Premise 2**: $q \to r$ **Conclusion**: $p \to r$

Thus, the argument converts to:
$((p \to q) \wedge (q \to r)) \to (p \to r)$

Argument type:

Valid or Invalid?

If you stay in, your roommate goes out.
If your roommate doesn't go out, s/he will finish their math homework.
Your roommate doesn't finish their math homework.
Therefore, you do not stay in.

Valid or Invalid?


```
Either this milk has soured, or I have the flu.
The milk has not soured.
--------------------------------------
I have the flu.

p =

q =
```

Premise 1: $p \vee q$ **Premise 2**: $\sim p$ **Conclusion**: q

Thus, the argument converts to: $((p \vee q) \wedge \sim p) \rightarrow q$

Do I have the flu?

Argument type:

Valid or Invalid?

```
If it purrs, it's a cat.
If it's a cat, I'm allergic to it.
-----------------------------------
If it purrs, I'm allergic to it.

p =

q =

r =
```

Argument:

Argument type:

Valid or Invalid?

If you use binoculars, then you get a glimpse of the comet.
If you get a glimpse of the comet, then you'll be amazed.

If you use binoculars, then you'll be amazed.

Valid or Invalid?

If he buys another toy, his toy chest will overflow.
His toy chest overflows.

He bought another toy.

Valid or Invalid?

If Ursula plays, the opponent loses.
The opponent does not lose.

Ursula does not play.

Valid or Invalid?

If we evolved a race of Isaac Newtons, that would be progress. (A. Huxley)

We have not evolved a race of Isaac Newtons.

That is not progress.

Valid or Invalid?

Alison pumps iron or Tom jogs.
Tom doesn't jog.

Alison pumps iron.

Valid or Invalid?

Jeff loves to play golf. If Joan likes to sew, then Jeff does not love to play golf. If Joan does not like to sew, then Brad sings in the choir. Therefore, Brad sings in the choir.

Valid or Invalid?

If the Bobble head doll craze continues, then Beanie Babies will remain popular. Barbie dolls continue to be favorites or Beanie Babies will remain popular. Barbie dolls do not continue to be favorites. Therefore, the Bobble head doll craze does not continue.

Valid or Invalid?

If Jerry is a DJ, then he lives in Lexington. He lives in Lexington and is a history buff. Therefore, if Jerry is not a history buff, then he is not a DJ.

Valid or Invalid?

If I've got you under my skin, then you are deep in the heart of me. If you are deep in the heart of me, then you are not really a part of me. You are deep in the heart of me, or you are really a part of me. Therefore, if I've got you under my skin, then you are really a part of me.

Determine a Valid Conclusion, If Possible

It is either day or night.
If it is daytime, then the squirrels are scurrying.
It is not nighttime.

Determine a Valid Conclusion, If Possible

If it is cold, you wear a coat.
If you don't wear a coat, you are dashing.
You aren't dashing.

Formal Proofs

To prove an argument is valid or the conclusion follows logically from the hypotheses:

- **Assume** the hypotheses are true
- Use the rules of inference and logical equivalences to determine that the conclusion is true.

Example

Consider the following logical argument:

If horses fly or cows eat artichokes, then the mosquito is the national bird. If the mosquito is the national bird, then peanut butter tastes good on hot dogs. But peanut butter tastes terrible on hot dogs. Therefore, cows don't eat artichokes.

Assign propositional variables to the component propositions in the argument:

H	Horses fly
C	Cows eat artichokes
M	The mosquito is the national bird
P	Peanut butter tastes good on hot dogs

Represent the formal argument using the variables:

1. $(H \vee C) \to M$
2. $M \to P$
3. $\neg P$

$\therefore \neg C$

Use Hypotheses & Rules of Inference

The proof:

(1)	$(H \vee C) \rightarrow M$	Hypothesis 1
(2)	$M \rightarrow P$	Hypothesis 2
(3)	$(H \vee C) \rightarrow P$	steps 1 & 2 and Hypothetical Syll
(4)	$\neg P$	Hypothesis 3
(5)	$\neg(H \vee C)$	steps 3 & 4, and Modus Tollens
(6)	$\neg H \wedge \neg C$	step 5 and DeMorgan
(7)	$\neg C \wedge \neg H$	step 6 and commutivity of $\wedge$
(8)	$\neg C$	step 7 and simplification

Q.E.D.

Methods of Proof

- We wish to establish the truth of the 'theorem': $p \to q$
- p may be a conjunction of other hypotheses
- $p \to q$ is a **conjecture** until a proof is produced

Trivial Proof

If we know q is true, then $p \to q$ is trivially true, regardless of the truth of p, since ($\texttt{anything} \to \textbf{T}$) is always true.

Example:

If it's raining today, then the empty set is a subset of every set.

The assertion is **trivially** true (since the empty set is a subset of every set).

Vacuous Proof

If we know one of the hypotheses in p is false, then $p \to q$ is **vacuously** true, Since (**F** $\to$ `anything`) is true.

Example:

If I am both rich and poor, then hurricane Fran was a mild breeze.

This has the form: $(p \wedge \neg p) \to q$

and the hypotheses form a **contradiction**.

Hence, q follows from the hypotheses vacuously.

Direct Proof

- Assumes the hypotheses are true
- Uses the rules of inference, axioms, and any logical equivalences to establish the truth of the conclusion.

[Example: The *Cows don't eat artichokes* proof previously.]

Another Example

Theorem: If $6x + 9y = 101$, then x or y is not an integer.

Proof (direct):

- Assume $6x + 9y = 101$ is true.
- Then, from the rules of algebra, $3(2x + 3y) = 101$
- But, $\frac{101}{3}$ is not an integer, so it must be the case that one of x or y is not an integer (maybe both)
- $\therefore$ one of x or y must not be an integer

Q.E.D.

Indirect Proof

A direct proof of the **contrapositive**:

- Assumes the conclusion of $p \to q$ is false (i.e., $\neg q$ is true)
- Uses the rules of inference, axioms, and any logical equivalences to establish the premise p is false.

Note: in order to show that a conjunction of hypotheses is false, it suffices to show just one of the hypotheses is false.

Example

A **perfect** number is one which is the sum of all its divisors, except itself. For example, 6 is perfect since $1 + 2 + 3 = 6$. So is 28.

Theorem: A perfect number is not a prime.

Proof (indirect):

- We assume the number p is prime, and show it is not perfect.
- The only divisors of a prime are 1 and itself.
- Hence the sum of the divisors less than p is 1, which is not equal to p.
- $\therefore$ p cannot be perfect.

Q.E.D.

Proof by Contradiction or Reductio Ad Absurdum

- Assume the conclusion q is false
- Derive a contradiction, usually of the form $p \wedge \neg p$ which establishes $\neg q \rightarrow \mathit{False}$

The contrapositive of this assertion is $\mathit{True} \rightarrow q$, from which it follows that q must be true.

Example

Theorem: There is no largest prime number.

(Note: there are no formal hypotheses here.)

Proof (by contradiction):

- Assume the conclusion, **there is no largest prime number** is false.
- There is a largest prime number, call it P.
- Hence, the set of all primes lie between 1 and P.
- Form the product of these primes:
  $R = 2 \times 3 \times 5 \times 7 \times \cdots \times P$
- But $R + 1$ is a prime larger than P. (Why?)
- This contradicts the assumption that there is a largest prime.

Q.E.D.

Formal Structure of This Proof

- Let p be the assertion that there is no largest prime.
- Let q be the assertion that P is the largest prime.
- Assume $\neg p$ is true.
- Then (for some P), q is true, so $\neg p \rightarrow q$ is true.
- Construct a prime greater than P, so $q \rightarrow \neg q$
- Apply hypothetical syllogism to get $\neg p \rightarrow \neg q$

From two applications of **modus ponens**, we conclude that q is true, and $\neg q$ is true, so by conjunction, $\neg q \wedge q$ or a contradiction is true.

Hence, the assumption must be false, and the theorem is true.

Proof By Cases

- Break the premise of $p \to q$ into an equivalent disjunction of the form:
$$p_1 \vee p_2 \vee \cdots \vee p_n$$
- Then use the tautology:

$$(p_1 \to q) \wedge (p_2 \to q) \wedge \cdots \wedge (p_n \to q) \iff [(p_1 \vee p_2 \vee \cdots \vee p_n) \to q]$$

- Each of the implications $p_i \to q$ is a **case**.
- You must convince the reader that the cases are inclusive, i.e., they exhaust all possibilities.
- Establish all implications.

Example

Let $\otimes$ be the operation **max** on the set of integers:

$$\text{if } a \geq b \text{ then } a \otimes b = max\{a, b\} = a = b \otimes a$$

Theorem. The operation $\otimes$ is associative.

$$\text{For all } a, b, c\text{: } (a \otimes b) \otimes c = a \otimes (b \otimes c).$$

Proof.

- Let a, b, and c be unique, arbitrary integers.
- Then one of the following six cases must hold (i.e., are exhaustive):
  1. $a \geq b \geq c$
  2. $a \geq c \geq b$
  3. $b \geq a \geq c$
  4. $b \geq c \geq a$
  5. $c \geq a \geq b$
  6. $c \geq b \geq a$

Case I

- $a \otimes b = a$, $a \otimes c = a$, and $b \otimes c = b$.
- Hence, $(a \otimes b) \otimes c = a = a \otimes (b \otimes c)$.
- Therefore the equality holds for the first case.
- The proofs of the remaining cases are similar (and are left for the student).

Q.E.D.