# Mat 2345

Week 4

Fall 2013

# Student Responsibilities — Week 4



- **Reading**: Textbook, Section 2.1–2.3
- **Attendance**: Strongly Encouraged

- **Week 4 Overview**
  - 2.1 Sets
  - 2.2 Set Operations
  - 2.3 Functions

# 2.1 Sets



- **Set**: an **unordered** collection or group of objects, which are said to be **elements**, or **members** of the set
- A set is said to **contain** its elements
- There must be an underlying **Universal Set,** $U$, either specifically stated or understood

# Notation used to specify sets



- **list the elements between braces**; listing an object more than once does *not* change the set—ordering means nothing.

$$S = \{a, b, c, d\} = \{b, c, a, d, d\}$$

- **Set builder notation** – specify by predicate; here, $S$ contains all elements from $U$ which make the predicate $P$ TRUE

$$S = \{ x \mid P(x) \}$$

- **brace notation with ellipses**; here, the negative integers:

$$S = \{\ldots, -3, -2, -1\}$$

# Common Universal Sets



- $\mathbb{R}$ — Real Numbers
- $\mathbb{N}$ — Natural Numbers: $\{0, 1, 2, 3, \ldots\}$
- $\mathbb{Z}$ — Integers: $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
- $\mathbb{Z}^+$ — Positive Integers
- $\mathbb{Q}$ — Rational Numbers: $\{\ \frac{p}{q} \mid p, q \in \mathbb{Z} \land q \neq 0\}$

# Notation



- $x \in S$ — $x$ is a member of $S$, or $x$ is an element of $S$
- $x \notin S$ — $x$ is not an element of $S$

## Set Equality — Definition #1

- Two sets are **equal** if and only if they have the same elements.
- That is, if $A$ and $B$ are sets, then $A$ and $B$ are equal if and only if $\forall x[x \in A \leftrightarrow x \in B]$.
- We write $A = B$ if $A$ and $B$ are equal sets.

# Subsets



- **Subset**: Let $A$ and $B$ be sets. Then
  $$A \subseteq B \Leftrightarrow \forall x \; [\, x \in A \rightarrow x \in B \,]$$

- **Empty, void,** or **Null Set**: $\varnothing$ is the set with no members
  - the assertion $x \in \varnothing$ is **always** FALSE, thus:
    $\forall x \; [\, x \in \varnothing \rightarrow x \in B]$ is always (vacuously) TRUE,
    and **therefore** $\varnothing$ is a **subset of every set**
  - **Note**: a set $B$ is always a subset of itself: $B \subseteq B$

- **Proper subset**: $A \subset B$ if $A \subseteq B$, but $A \neq B$

# Power Set



- **Power Set**: $\mathscr{P}(A)$ is the set of *all* possible subsets of the set $A$
- If $A = \{a, b\}$, then
  $$\mathscr{P}(A) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$$
- What is the power set of the set $B = \{0, 1, 2\}$?
- How many elements would $\mathscr{P}(\{a, b, c, d\})$ have?

# Cardinality

- **Cardinality**: $|A|$ is the number of distinct elements in $A$
- If the cardinality is a natural number (in $\mathbb{N}$), then the set is called **finite**; otherwise, it's called **infinite**
- **Example**: Let $A = \{a, b\}$
  - $|A| = |\{a, b\}| = 2$
  - $|\mathscr{P}(A)| = |\mathscr{P}(\{a, b\})| = 4$
  - $A$ is finite, and so is $\mathscr{P}(A)$
- **Note$_1$**: $|A| = n \rightarrow |\mathscr{P}(A)| = 2^n$
- **Note$_2$**: $\mathbb{N}$ is infinite since $|\mathbb{N}|$ is not a natural number — it is called a **transfinite cardinal number**
- **Note$_3$**: Sets can be both **members** and **subsets** of other sets

# Example



- Let $A = \{\varnothing, \{\varnothing\}\}$
  - A has two elements and hence four subsets:
    $$\varnothing, \quad \{\varnothing\}, \quad \{\{\varnothing\}\}, \quad \{\varnothing, \{\varnothing\}\}$$
  - Note that $\varnothing$ is both a member of $A$ and a subset of $A$

- **Russell's Paradox**: Let $S$ be the set of all sets which are not members of themselves.
  Is $S$ a member of itself or not?

- **The Paradox of the Barber of Seville**: The (male) barber of Seville shaves all and only men who do not shave themselves. Who shaves the barber of Seville?

# Cartesian Product

- **Cartesian Product of A with B**: $A \times B$ is the set of **ordered** pairs: $\{ < a, b > \mid a \in A \wedge b \in B\}$
- **Notation**: $\Pi_{i=1}^{n} A_i = \{< a_1, a_2, \ldots, a_n > \mid a_i \in A_i\}$, an **n–tuple**
- The Cartesian product of any set with $\varnothing$ is $\varnothing$ — why?
- **Example 1**. Let $A = \{a, b\}$ and $B = \{1, 2, 3\}$

  $A \times B = \{< a, 1 >, < a, 2 >, < a, 3 >, < b, 1 >, < b, 2 >, < b, 3 >\}$

  What is $B \times A$?

  If $|A| = m$ and $|B| = n$, what is $|A \times B|$?

- The Cartesian product of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$ is the set of ordered $n$–tuples $< a_1, a_2, \ldots, a_n >$, where $a_i \in A_i$ for $1 \leq i \leq n$.
- $A_1 \times A_2 \times \cdots \times A_n =$
$$\{< a_1, a_2, \ldots, a_n > | a_i \in A_i, i = 1, 2, \ldots, n\}$$
- If $A = \{a, b\}$ and $B = \{1, 2, 3\}$, what is $A \times B \times A$?

# Quantifiers

The **Universe of Discourse**, also known as the **Domain of Discourse**, is often referred to simply as the **domain**.

The domain specifies the possible values of our variables.

We can use quantifiers to restrict the domain

- $\forall x \in S[P(x)]$ denotes $\forall x[x \in S \rightarrow P(x)]$

  **Ex:** $\forall x \in \mathbb{R}[x^2 \geq 0]$ means:
  *for every real number $x$, $x^2$ is non-negative*

  **Note**: The meaning of the universal quantifier changes when we change the domain.

- $\exists x \in S[P(x)]$ denotes $\exists x[x \in S \wedge P(x)]$

  **Ex:** $\exists x \in \mathbb{Z}[x^2 = 1]$ means:
  *there exists an integer $x$ such that $x^2 = 1$*

# Truth Sets

- Let $P$ be a predicate and $D$ a domain. The **Truth Set** of $P$ is the set of elements $x \in D \ni P(x)$ is true.
- The truth set of $P(x)$ is denoted: $\{x \in D | P(x)\}$
- Assume the domain is the set of integers. What are the truth sets:
  - $P = \{x \in \mathbb{Z} \mid |x| = 1\}$ Truth Set:
  - $Q = \{x \in \mathbb{Z} \mid x^2 = 2\}$ Truth Set:
  - $R = \{x \in \mathbb{Z} \mid |x| = x\}$ Truth Set:
- **Note**: $\forall x P(x)$ is true over the domain $U$ IFF the truth set of $P$ is $U$.
- **Note**: $\exists x P(x)$ is true over the domain $U$ IFF the truth set of $P \neq \varnothing$.

# 2.2 Set Operations

- **Boolean Algebra**: an algebraic system, instances of which are **propositional calculus** and **set theory**.
- The operators in set theory are defined in terms of the corresponding operator in propositional calculus.
- As before, there must be a universe, $U$, and all sets are assumed to be subsets of $U$.

# Equality of Sets



By a previous logical equivalence, we have:

$$A = B$$

IFF

$$\forall x\, [(x \in A \to x \in B) \wedge (x \in B \to x \in A)]$$

— **or** another **definition** —

$$A = B \quad \text{IFF} \quad A \subseteq B \quad \text{and} \quad B \subseteq A$$

# Set Operations



- **Union** of $A$ and $B$, denoted $A \cup B$, is the set
$$\{x \mid x \in A \vee x \in B\}$$

- **Intersection** of $A$ and $B$, denoted $A \cap B$, is the set
$$\{x \mid x \in A \wedge x \in B\}$$
If the intersection is void, $A$ and $B$ are said to be **disjoint**

- **Complement** of $A$, denoted $\overline{A}$, is the set
$$\{x \mid \neg(x \in A)\} = \{x \mid x \notin A\}$$

# More Set Operations



- **Difference** of $A$ and $B$, or the **complement of $B$ relative to $A$**, denoted $A - B$, is the set

$$A \cap \overline{B}$$

Note: The absolute complement of $A$ is $U - A$

- **Symmetric Difference** of $A$ and $B$, denoted $A \oplus B$, is the set

$$(A - B) \cup (B - A)$$

# Examples



$$U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

$$A = \{1, 2, 3, 4, 5\} \quad \text{and} \quad B = \{4, 5, 6, 7, 8\}$$

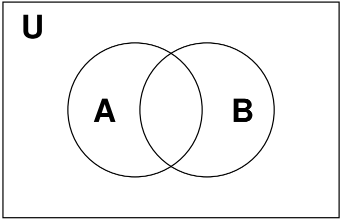
- $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$
- $A \cap B = \{4, 5\}$
- $\overline{A} = \{0, 6, 7, 8, 9, 10\}$
- $\overline{B} = \{0, 1, 2, 3, 9, 10\}$
- $A - B = \{1, 2, 3\}$
- $B - A = \{6, 7, 8\}$
- $A \oplus B = \{1, 2, 3, 6, 7, 8\}$

# Venn Diagrams

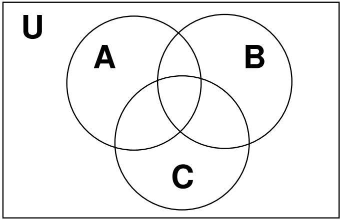


Venn Diagrams are a useful geometric visualization tool for 3 or fewer sets.

- The Universal set $U$ is a rectangular box
- Each set is represented by a circle and its interior
- All possible combinations of the sets must be represented
- Shade the appropriate region to represent the given set operation



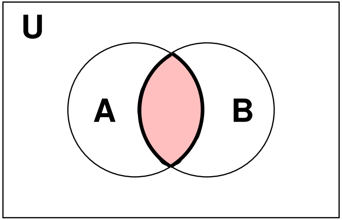

For 2 sets



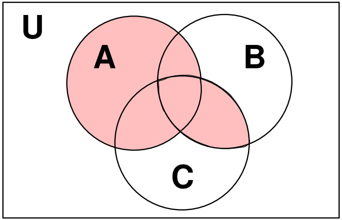

For 3 sets

# Examples






$A \cap B$




$A \cup (C \cap B)$

# Set Identities



**Set identities correspond to the logical equivalences.**

**Example**

The complement of the union is the intersection of the complements:

$$\overline{A \cup B} \;=\; \overline{A} \cap \overline{B}$$

To prove this statement, we would need to show:

$$\forall x \; [x \in \overline{A \cup B} \;\leftrightarrow\; x \in \overline{A} \cap \overline{B}]$$

**To show two sets are equal**:

- we can show for all $x$ that $x$ is a member of one set IFF it is a member of the other, or
- show that each is a subset of the other

# Universal Instantiation

We now apply an important **rule of inference** called

*Universal Instantiation*

In a proof, we can eliminate the universal quantifier which binds a variable if we do not assume anything about the variable other than it is an arbitrary member of the Universe. We can then treat the resulting predicate as a proposition.

We say, **"Let $x$ be arbitrary."** Then we can treat the predicates as propositions.

| | | Assertion | Reason |
|---|---|---|---|
| $x \in \overline{A \cup B}$ | $\Leftrightarrow$ | $x \notin (A \cup B)$ | Defn of complement |
| | $\Leftrightarrow$ | $\neg[x \in (A \cup B)]$ | Defn of $\notin$ |
| | $\Leftrightarrow$ | $\neg[(x \in A) \vee (x \in B)]$ | Defn of union |
| | $\Leftrightarrow$ | $\neg(x \in A) \wedge \neg(x \in B)$ | DeMorgan's Laws |
| | $\Leftrightarrow$ | $(x \notin A) \wedge (x \notin B)$ | Defn of $\notin$ |
| | $\Leftrightarrow$ | $(x \in \overline{A}) \wedge (x \in \overline{B})$ | Defn of complement |
| | $\Leftrightarrow$ | $x \in (\overline{A} \cap \overline{B})$ | Defn of intersection |

Hence, $x \in \overline{A \cup B} \leftrightarrow x \in \overline{A} \cap \overline{B}$ is a **tautology** since

- $x$ was arbitrary, and
- we have used only logically equivalent assertions and definitions

# Universal Generalization

We can apply another rule of inference

*Universal Generalization*

We can apply a universal quantifier to bind a variable if we have shown the predicate to be true for all values of the variable in the Universe

and claim the assertion is true for all $x$, i.e.,

$$\forall x[x \in \overline{A \cup B} \;\leftrightarrow\; x \in \overline{A} \cap \overline{B}]$$

Q.E.D. — an abbreviation for the Latin phrase *"Quod Erat Demonstrandum"* – "which was to be demonstrated" – used to signal the end of a proof.

# Alternative Identity





Note: as an alternative which might be easier in some cases, use the identity:

$$A = B \Leftrightarrow [A \subseteq B \text{ and } B \subseteq A]$$

**Example**

Show $A \cap (B - A) = \emptyset$

The empty set is a subset of every set. Hence,

$$A \cap (B - A) \supseteq \emptyset$$

Therefore, it suffices to show

$$A \cap (B - A) \subseteq \emptyset$$

or

$$\forall x[x \in A \cap (B - A) \rightarrow x \in \emptyset]$$

So, as before, we say **"let $x$ be arbitrary"**

Now we need to **show**

$$x \in A \cap (B - A) \ \rightarrow \ x \in \emptyset$$

is a **tautology**.

But the **consequent** is always **false**.

Therefore, the **antecedent** (or premise) must also be **false**.

We proceed by applying the definitions:

| | Assertion | Reason |
|---|---|---|
| $x \in A \cap (B - A)$ | $\Leftrightarrow (x \in A) \wedge [x \in (B - A)]$ | Defn of intersection |
| | $\Leftrightarrow (x \in A) \wedge [(x \in B) \wedge (x \notin A)]$ | Defn of difference |
| | $\Leftrightarrow [(x \in A) \wedge (x \notin A)] \wedge (x \in B)$ | Comm Prop of AND |
| | $\Leftrightarrow F \wedge (x \in B)$ | Table 6 in textbook |
| | $\Leftrightarrow F$ | Domination |

Hence, because $P \wedge \neg P$ is always false, the implication is a tautology. The result follows by Universal Generalization. Q.E.D.

# Indexed Collections



Let $A_1, A_2, \ldots, A_n$ be an indexed collection of sets.
Union and intersection are associative (because AND and OR are), we have:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

and

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$

**Examples**

Let $\quad A_i = [\ i, \infty), \quad 1 \leq i < \infty$

$$\bigcup_{i=1}^{n} A_i = [1, \infty)$$

$$\bigcap_{i=1}^{n} A_i = [n, \infty)$$

# 2.3 Functions



- **Function**: Let $A$ and $B$ be sets. Then a function (mapping, map) $f$ from $A$ to $B$, denoted $f : A \rightarrow B$, is a **subset** of $A \times B$ such that

$$\forall x \, [x \in A \rightarrow \exists y \, [\, y \in B \;\wedge\; < x, y > \;\in f \,]]$$

and

$$[\, < x, y_1 > \;\in\; f \;\wedge\; < x, y_2 > \;\in\; f \,] \rightarrow y_1 \;=\; y_2$$

- **Note**: $f$ associates with each $x \in A$ **one and only one** $y \in B$.

- $A$ is called the **domain** of $f$

- $B$ is called the **codomain** of $f$

- If $F(x) = y$:
  - $y$ is called the **image** of $x$ under $f$
  - $x$ is called a **preimage** of $y$

- **Note**: there may be more than one preimage of $y$, but there is only **one** image of $x$.

- The **range** of $f$ is the set of all images of points in $A$ under $f$; it is denoted by $f(A)$

# Injections, Surjections, and Bijections



Let $f$ be a function from $A$ to $B$

- **Injection**: $f$ is **one–to–one** (denoted 1–1) or **Injective** if preimages are unique

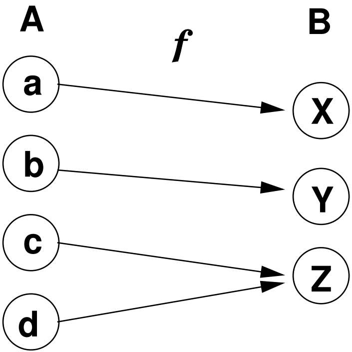
  Note: this means that if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$

- **Surjection**: $f$ is **onto** or **surjective** if every $y$ in $B$ has a preimage

  Note: this means that for every $y$ in $B$ there must be an $x$ in $A$ such that $f(x) = y$

- **Bijection**: $f$ is **bijective** if it is surjective and injective, in other words, **1–1 and onto**.

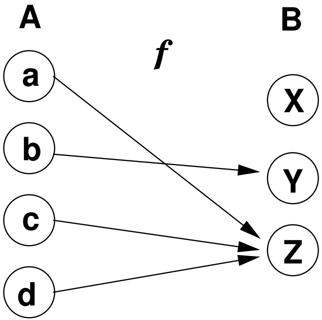
# Example I



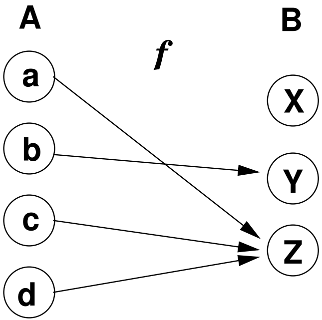

1. Is this an injection?
2. Is this a surjection?
3. Is this a bijection?
4. How do we determine the answers?

# Example II

If $S$ is a subset of $A$, then $f(S) = \{f(s) | s \in S\}$

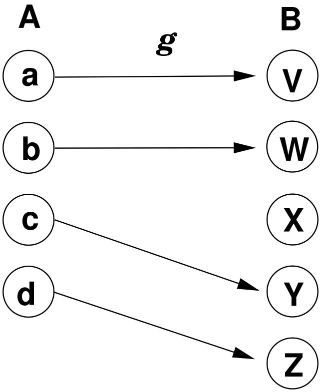



- $f(a) = Z$; the image of $d$ is $Z$
- $f(\{a, d\}) = \{Z\}$
- the domain of $f$ is $A = \{a, b, c, d\}$
- The range of $f$ is $f(A) = \{Y, Z\}$




- the codomain is $B = \{X, Y, Z\}$
- the preimage of $Y$ is $b$
- the preimages of $Z$ are $a$, $c$, and $d$
- $f(\{a, b\}) = \{Y, Z\}$

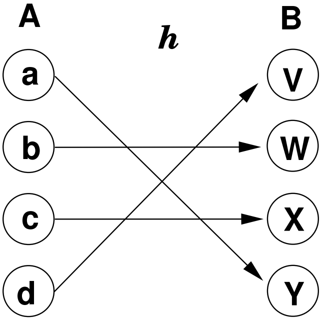
# Example III




1. Is this an injection?
2. Is this a surjection?
3. Is this a bijection?

# Example IV






1. Is this an injection?
2. Is this a surjection?
3. Is this a bijection?

# Cardinality



- **Note**: whenever there is a bijection from $A$ to $B$, the two sets must have the **same number of elements** or the **same cardinality**
- That will become our definition, especially for infinite sets.

Let $A = B = \mathbb{R}$, the reals
Determine which are injections, surjections, bijections:

| Function | Injection? 1—1 | Surjection? onto | Bijection? |
|---|---|---|---|
| $f(x) = x$ | | | |
| $f(x) = x^2$ | | | |
| $f(x) = x^3$ | | | |
| $f(x) = \|x\|$ | | | |

Let $E$ be the set of even nonnegative integers, $\{0, 2, 4, 6, \ldots\}$

Then there is a bijection $f$ from $\mathbb{N}$ to $E$, the even nonnegative integers, defined by:

$$f(x) = 2x$$

Hence, the set of even nonnegative integers has the **same cardinality** as the set of natural numbers. . .

**OH, NOES! IT CAN'T BE. . . E is only half as big!!**

**(But it's TRUE.)**

# Inverse Functions



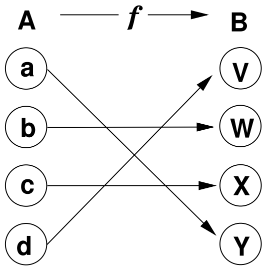
- **Inverse Function**: Let $f$ be a bijection from $A$ to $B$. Then the **inverse** of $f$, denoted $f^{-1}$, is the function from $B$ to $A$ defined as:

$$f^{-1}(y) = x \quad \text{IFF} \quad f(x) = y$$

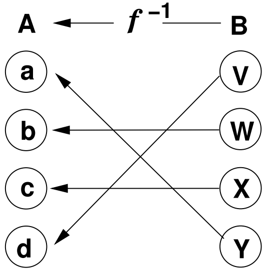
- Note: no inverse exists unless $f$ is a bijection.

# Example

Let $f$ be defined by the diagram:




Then $f^{-1}$ is defined as:



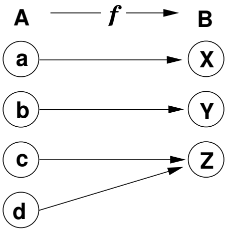
# Inverse Applied to a Subset



- **Inverse Function over Subsets**: Let $S$ be a subset of $B$. Then

$$f^{-1}(S) \;=\; \{x \mid f(x) \in S\}$$

- Example: Let $f$ be the following function –



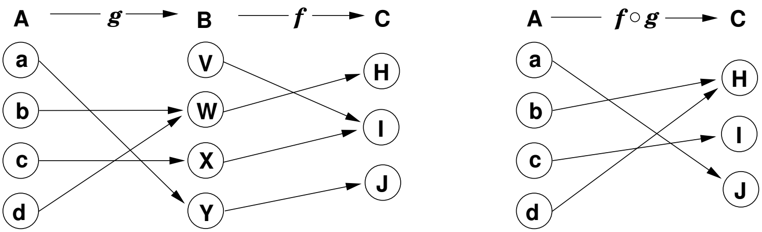

$$f^{-1}(\{X, Y\}) \;=\; \{a, b\}$$

# Composition

- **Composition**: Let $f : B \to C$, $g : A \to B$.
  The **composition of $f$ with $g$**, denoted $f \circ g$, is the function from $A$ to $C$ defined by

$$f \circ g(x) = f(g(x))$$

- Example:



# Other Examples



Let $f(x) = x^2$ and $g(x) = 2x + 1$

$$\begin{aligned} f \circ g(x) &= f(g(x)) \\ &= f(2x+1) \\ &= (2x+1)^2 \\ &= 4x^2 + 4x + 1 \end{aligned}$$

$$\begin{aligned} g \circ f(x) &= g(f(x)) \\ &= g(x^2) \\ &= 2x^2 + 1 \end{aligned}$$

## Discussion

- Suppose $f : B \to C$, $g : A \to B$, and $f \circ g$ is injective
- What can we say about $f$ and $g$?
- Using the definition of **injective**, we know that if $a \neq b$, then $f(g(a)) \neq f(g(b))$, since the composition is injective
- Since $f$ is a function, it cannot be the case that $g(a) = g(b)$, since $f$ would have two different images for the same point.
- Hence, $g(a) \neq g(b)$
- It follows that $g$ must be an injection
- However, $f$ need not be an injection. . . how could you show this? (counterexample)

# FLOOR and CEILING Functions



- **Floor**: The FLOOR function, denoted

  $$f(x) = \lfloor x \rfloor \quad \text{or} \quad f(x) = \text{FLOOR}(x)$$

  is the largest integer less than or equal to $x$.

- **Ceiling**: The CEILING function, denoted

  $$f(x) = \lceil x \rceil \quad \text{or} \quad f(x) = \text{CEILING}(x)$$

  is the smallest integer greater than or equal to $x$.

- Examples: $\lfloor 3.5 \rfloor = 3, \quad \lceil 3.5 \rceil = 4$

- Note: The FLOOR function is equivalent to **truncation** for positive numbers