Mat 2345 — Discrete Math

Week 13

Fall 2013

Student Responsibilities — Week 13

- **Reading**: Textbook, Section 7.1 & 7.2
- **Assignments**: Sec 7.1, 7.2
- **Attendance**: De–Lightfully Encouraged

Week 12 Overview

- Sec 7.1 Recurrence Relations
- Sec 7.2 Solving Linear Recurrence Relations

- A **recursive definition** of a sequence specifies one or more initial terms plus a rule for determining subsequent terms from those that precede them.

- A **recurrence relation** for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

- The **initial conditions** for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

Recurrence Relations, Cont.

- A sequence is called a **solution** of a recurrence relation if its terms satisfy the recurrence relation.

- A **Recurrence Relation** is a way to define a function by an expression involving the same function.

Modeling with Recurrence Relations – Rabbits

- A pair of rabbits (one of each sex) is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair produces another pair each month.

- **Fibonacci Numbers** (Pairs of Rabbits)
  `F(0) = 1,   F(1) = 1,`
  `F(n) = F(n-1) + F(n-2)`

- If we wish to compute the 120^{th} Fibonacci Number, `F(120)`, we could compute `F(0), F(1), F(2), ...F(118)`, and `F(119)` to arrive at `F(120)`.

- Thus, to compute `F(k)` in this manner would take `k` steps.

Fibonacci Closed Form Expression

- It would be more convenient, not to mention more efficient, to have an **explicit** or **closed form** expression to compute `F(n)`.

- Actually, for Fibonacci numbers, it's:

$$F(n) \; = \; \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n$$

$$\forall \; natural \; numbers \; n \geq 1$$

Modeling with Recurrence Relations – Compound Interest

- General problem: A person makes a deposit (principle) into a savings account which yields a yearly interest rate, compounded annually. How much will be in the account after 30 years?
- Let P_n represent the amount in the account after n years.
- Since P_n will equal the amount after $n-1$ years plus interest, the sequence $\{P_n\}$ satisfies the recurrence relation:

$$P_n = P_{n-1} + rP_{n-1} = (1+r)P_{n-1}$$

Recurrence Relations – Compound Interest

- We can use an iterative approach to find a formula for P_n:

$$\begin{aligned}
P_1 &= (1+r)P_0 & & \\
P_2 &= (1+r)P_1 &=& (1+r)^2 P_0 \\
P_3 &= (1+r)P_2 &=& (1+r)^3 P_0 \\
&\vdots & & \\
P_n &= (1+r)P_{n-1} &=& (1+r)^n P_0
\end{aligned}$$

- Let's assume \$10,000 was deposited at 11% interest rate, compounded annually, for 30 years.
- Then $P_{30} \; = \; (1.11)^{30} 10,000 \; = \; \$228,922.97$
- See other examples of modeling with RR in textbook.

- Recurrence relations which express the terms of a sequence as a **linear combination of previous terms** can be explicitly solved in a systematic way.

- **Definition** A **linear homogeneous recurrence relation of degree k with constant coefficients** is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$

- **Linear**: the right–hand side is a sum of multiples of the previous terms of the sequence.
- **Homogeneous**: no terms occur that are **not** multiples of the a_j's
- **Constant Coefficients**: the coefficients of all the terms of the sequence are constants (rather than functions dependent on n)
- **Degree**: is k because a_n is expressed in terms of the previous k terms of the sequence.

A sequence satisfying the recurrence relation in the definition is uniquely determined by this recurrence relation and the k initial conditions:

$$a_0 = C_0, \qquad a_1 = C_1, \qquad \ldots, \qquad a_{k-1} = C_{k-1},$$

Examples of linear homogeneous recurrence relations:

$P_n = 3P_{n-1}$	degree one
$f_n = f_{n-1} + f_{n-2}$	degree two
$a_n = a_{n-5}$	degree five

Examples which are **not** linear homogeneous recurrence relations:

$a_n = a_{n-1} + a_{n-2}^2$	not linear
$H_n = 2H_{n-1} + 2$	not homogeneous
$B_n = nB_{n-5}$	doesn't have constant coefficient

Idea: look for solutions of the form $a_n = r^n$, where r is a constant.

Note: $a_n = r^n$ is a solution of the recurrence relation:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

if and only if

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \ldots + c_k r^{n-k}$$

Divide both sides of the previous equation by r^{n-k}, and subtract the right–hand side:

$$r^k \; - \; c_1 r^{k-1} \; - \; c_2 r^{k-2} \; - \; \ldots \; - \; c_{k-1} r \; - \; c_k \; = \; 0$$

This is the **characteristic equation** of the recurrence relation.

Note: The sequence $\{a_n\}$ with $a_n = r^n$ is a solution IFF r is a solution to the characteristic equation.

Characteristic Roots

The solutions of the characteristic equation are called the **characteristic roots** of the recurrence relation.

They can be used to create an explicit formula for all the solutions of the recurrence relation.

Characteristic Roots

Theorem 1. Let c_1 and c_2 be real numbers. Suppose that

$$r^2 - c_1 r - c_2 = 0$$

has two distinct roots, r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

if and only if

$$a_n = \propto_1 r_1^n + \propto_2 r_2^n$$

for $n = 0, 1, 2, \ldots,$ where $\propto_1$ and $\propto_2$ are constants

Solving Recurrence Relations, Example I

Let: $a_0 = 2$, $a_1 = 7$, **and** $a_n = a_{n-1} + 2a_{n-2}$

We see that $c_1 = 1$ and $c_2 = 2$

Characteristic Equation: $r^2 - r - 2 = 0$

Roots: $r = 2$ and $r = -1$

Thus, the sequence $\{a_n\}$ is a solution to the recurrence relation IFF

$$a_n = \propto_1 2^n + \propto_2 (-1)^n$$

for some constants $\propto_1$ and $\propto_2$

Solving Recurrence Relations, Example I — Cont.

From the initial conditions, it follows that:

$$\begin{array}{lllll} a_0 & = & 2 & = & \propto_1 (2^0) \; + \; \propto_2 (-1)^0 \\ a_1 & = & 7 & = & \propto_1 (2^1) \; + \; \propto_2 (-1)^1 \end{array}$$

Solving these two equations yields:

$$\propto_1 \; = \; 3 \quad \text{and} \quad \propto_2 \; = \; -1$$

Therefore, the **solution** to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with:

$$a_n \; = \; 3(2)^n \; - \; (-1)^n$$

Solving Recurrence Relations, Example II

Let: $F_0 = 0,\quad F_1 = 1$, **and** $\quad F_n = F_{n-1} + F_{n-2}$

We see that $c_1 = 1$ and $c_2 = 1$

Characteristic Equation: $\quad r^2 - r - 1 = 0$

Roots: $\quad r = \frac{1+\sqrt{5}}{2}$ and $r = \frac{1-\sqrt{5}}{2}$

Thus, it follows that the Fibonacci numbers are given by

$$F_n = \propto_1 \left(\tfrac{1+\sqrt{5}}{2}\right)^n + \propto_2 \left(\tfrac{1-\sqrt{5}}{2}\right)^n$$

for some constants $\propto_1$ and $\propto_2$

Solving Recurrence Relations, Example II — Cont.

From the initial conditions, it follows that:

$$\begin{array}{lllll} F_0 & = & 0 & = & \alpha_1 + \alpha_2 \\ F_1 & = & 1 & = & \alpha_1\left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right) \end{array}$$

Solving these two equations yields:

$$\alpha_1 = \frac{1}{\sqrt{5}} \quad \text{and} \quad \alpha_2 = -\frac{1}{\sqrt{5}}$$

Therefore, the **solution** to the recurrence relation and initial conditions is the sequence $\{F_n\}$ with:

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Solving Recurrence Relations, Example III

Let: $a_0 = 1$, $a_1 = 1$, **and** $a_n = 2a_{n-1} + 3a_{n-2}$

We see that $c_1 = 2$ and $c_2 = 3$

Characteristic Equation: $r^2 - 2r - 3 = 0$

Roots: $r = 3$ and $r = -1$

Thus, the sequence $\{a_n\}$ is a solution to the recurrence relation IFF

$$a_n = \propto_1 3^n + \propto_2 (-1)^n$$

for some constants $\propto_1$ and $\propto_2$

Solving Recurrence Relations, Example III — Cont.

From the initial conditions, it follows that:

$$\begin{aligned} a_0 &= 1 = \alpha_1 + \alpha_2 \\ a_1 &= 1 = \alpha_1\,(3) + \alpha_2\,(-1) \end{aligned}$$

Solving these two equations yields: $\alpha_1 = \frac{1}{2}$ and $\alpha_2 = \frac{1}{2}$

Therefore, the **solution** to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with:

$$a_n = \frac{1}{2}(3)^n + \frac{1}{2}(-1)^n$$

Solving Recurrence Relations, Example IV

Let: $a_0 = 1$, $a_1 = -2$, **and** $a_n = 5a_{n-1} - 6a_{n-2}$

We see that $c_1 = 5$ and $c_2 = -6$

Characteristic Equation: $r^2 - 5r + 6 = 0$

Roots: $r = 2$ and $r = 3$

Thus, the sequence $\{a_n\}$ is a solution to the recurrence relation IFF

$$a_n = \propto_1 2^n + \propto_2 3^n$$

for some constants $\propto_1$ and $\propto_2$

Solving Recurrence Relations, Example IV — Cont.

From the initial conditions, it follows that:

$$\begin{array}{lllll} a_0 & = & 1 & = & \propto_1 + \propto_2 \\ a_1 & = & \text{-}2 & = & \propto_1 (2) + \propto_2 (3) \end{array}$$

Solving these two equations yields:

$\propto_1 = 5$ and $\propto_2 = -4$

Therefore, the **solution** to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with:

$$a_n = 5(2)^n - 4(3)^n$$

Solving Recurrence Relations, Example V

Let: $a_0 = 0$, $a_1 = 1$, **and** $a_n = a_{n-1} + 6a_{n-2}$

We see that $c_1 = 1$ and $c_2 = 6$

Characteristic Equation: $r^2 - r - 6 = 0$

Roots: $r = 3$ and $r = -2$

Thus, the sequence $\{a_n\}$ is a solution to the recurrence relation IFF

$$a_n = \alpha_1 \, 3^n + \alpha_2 \, (-2)^n$$

for some constants α_1 and α_2

Solving Recurrence Relations, Example V

From the initial conditions, it follows that:

$$\begin{aligned} a_0 &= 0 = \propto_1 + \propto_2 \\ a_1 &= 1 = \propto_1 (3) + \propto_2 (-2) \end{aligned}$$

Solving these two equations yields:

$$\propto_1 = \frac{1}{5} \quad \text{and} \quad \propto_2 = -\frac{1}{5}$$

Therefore, the **solution** to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with:

$$a_n = \frac{1}{5}(3)^n - \frac{1}{5}(-2)^n$$

What To Do When There's Only One Root?

Theorem 1 **does not apply** when there is a **single** characteristic root of multiplicity two.

Theorem 2. Let c_1 and c_2 be real numbers. Suppose that

$$r^2 - c_1 r - c_2 = 0$$

has only one root, r_0. Then the sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

if and only if

$$a_n = \propto_1 r_0^n + \propto_2 n r_0^n$$

for $n = 0, 1, 2, \ldots,$ where $\propto_1$ and $\propto_2$ are constants

Notice the **extra factor** of n in the second term!

Single Root, Example I

Let: $a_0 = 1$, $a_1 = 6$, **and** $a_n = 6a_{n-1} - 9a_{n-2}$

We see that $c_1 = 6$ and $c_2 = -9$

Characteristic Equation: $r^2 - 6r + 9 = 0$

Root: $r = 3$ with multiplicity 2

Thus, the sequence $\{a_n\}$ is a solution to the recurrence relation IFF

$$a_n = \alpha_1 3^n + \alpha_2 n(3)^n$$

for some constants α_1 and α_2

Single Root, Example I — Cont.

From the initial conditions, it follows that:

$$\begin{aligned} a_0 &= 1 = \propto_1 \\ a_1 &= 6 = \propto_1 (3) + \propto_2 (3) \end{aligned}$$

Solving these two equations yields: $\propto_1 = 1$ and $\propto_2 = 1$

Therefore, the **solution** to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with:

$$a_n = (3)^n + n(3)^n$$

Single Root, Example II

Let: $a_0 = 1$, $a_1 = 3$, **and** $a_n = 4a_{n-1} - 4a_{n-2}$

We see that $c_1 = 4$ and $c_2 = -4$

Characteristic Equation: $r^2 - 4r + 4 = 0$

Root: $r = 2$ with multiplicity 2

Thus, the sequence $\{a_n\}$ is a solution to the recurrence relation IFF

$$a_n = \propto_1 2^n + \propto_2 n2^n$$

for some constants $\propto_1$ and $\propto_2$

Single Root, Example II — Cont.

From the initial conditions, it follows that:

$$\begin{aligned} a_0 &= 1 = \propto_1 \\ a_1 &= 3 = \propto_1 (2) + \propto_2 (2) \end{aligned}$$

Solving these two equations yields: $\propto_1 = 1$ and $\propto_2 = \frac{1}{2}$

Therefore, the **solution** to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with:

$$a_n = 2^n + \frac{1}{2} n 2^n = 2^n + n 2^{n-1}$$

Single Root, Example III

Let: $a_0 = 1$, $a_1 = 12$, **and** $a_n = 8a_{n-1} - 16a_{n-2}$

We see that $c_1 = 8$ and $c_2 = -16$

Characteristic Equation: $r^2 - 8r + 16 = 0$

Root: $r = 4$ with multiplicity 2

Thus, the sequence $\{a_n\}$ is a solution to the recurrence relation IFF

$$a_n = \propto_1 4^n + \propto_2 n4^n$$

for some constants $\propto_1$ and $\propto_2$

Single Root, Example III — Cont.

From the initial conditions, it follows that:

$$\begin{aligned} a_0 &= 1 &= \propto_1 \\ a_1 &= 12 &= \propto_1 (4) + \propto_2 (4) \end{aligned}$$

Solving these two equations yields: $\propto_1 = 1$ and $\propto_2 = 2$

Therefore, the **solution** to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with:

$$a_n = (4)^n + 2n(4)^n$$

Single Root, Example IV

Let: $a_0 = 2$, $a_1 = 5$, **and** $a_n = 2a_{n-1} - a_{n-2}$

We see that $c_1 = 2$ and $c_2 = -1$

Characteristic Equation: $r^2 - 2r + 1 = 0$

Root: $r = 1$ with multiplicity 2

Thus, the sequence $\{a_n\}$ is a solution to the recurrence relation IFF

$$a_n = \propto_1 1^n + \propto_2 n(1)^n$$

for some constants $\propto_1$ and $\propto_2$

Single Root, Example IV — Cont.

From the initial conditions, it follows that:

$$\begin{aligned} a_0 &= 2 = \propto_1 \\ a_1 &= 5 = \propto_1 (1) + \propto_2 (1) \end{aligned}$$

Solving these two equations yields: $\propto_1 = 2$ and $\propto_2 = 3$

Therefore, the **solution** to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with:

$$a_n = 2(1)^n + 3n(1)^n = 2 + 3n$$

Solving Recurrence Relations

Definition. A **linear homogeneous recurrence relation of degree k with constant coefficients** is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

Theorem 3. Let $c_1, \ c_2, \ \ldots, \ c_k$ be real numbers. Suppose that the characteristic equation

$$r^k \ - \ c_1 r^{k-1} \ - \ \ldots \ - \ c_k \ = \ 0$$

has k distinct roots, $r_1, \ r_2, \ \ldots, \ r_k$. Then the sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n \ = \ c_1 a_{n-1} \ + \ c_2 a_{n-2} \ + \ \ldots \ + \ c_k a_{n-k}$$

if and only if

$$a_n \ = \ \propto_1 r_1^n \ + \ \propto_2 r_2^n \ + \ \ldots \ + \ \propto_k r_k^n$$

for $n = 0, 1, 2, \ldots,$ where $\propto_1, \ \propto_2, \ \ldots, \ \propto_k$ are constants

Multiple Distinct Roots, Example I

Let: $a_0 = 2$, $a_1 = 5$, $a_2 = 15$, **and**
$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$

We see that $c_1 = 6$, $c_2 = -11$, and $c_3 = 6$

Characteristic Equation:
$r^3 - 6r^2 + 11r - 6 = (r-1)(r-2)(r-3) = 0$

Roots: $r = 1$, $r = 2$, and $r = 3$

Thus, the sequence $\{a_n\}$ is a solution to the recurrence relation IFF

$$a_n = \propto_1 1^n + \propto_2 2^n + \propto_3 3^n$$

for some constants $\propto_1$, $\propto_2$, and $\propto_3$

Multiple Distinct Roots, Example I — Cont.

From the initial conditions, it follows that:

$$\begin{array}{rcrcl} a_0 & = & 2 & = & \propto_1 + \propto_2 + \propto_3 \\ a_1 & = & 5 & = & \propto_1 + \propto_2 (2) + \propto_3 (3) \\ a_2 & = & 15 & = & \propto_1 + \propto_2 (4) + \propto_3 (9) \end{array}$$

Solving: $\propto_1 = 1$, $\propto_2 = -1$, and $\propto_3 = 2$

Therefore, the **solution** to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with:

$$a_n = 1 - 2^n + 2(3)^n.$$

Multiple Distinct Roots, Example II

Let: $a_0 = 4$, $a_1 = -9$, $a_2 = -9$, **and** $a_n = 4a_{n-1} - a_{n-2} - 6a_{n-3}$

We see that $c_1 = 4$, $c_2 = -1$, and $c_3 = -6$

Characteristic Equation:
$r^3 - 4r^2 + r + 6 = (r+1)(r-2)(r-3) = 0$

Roots: $r = -1$, $r = 2$, and $r = 3$

Thus, the sequence $\{a_n\}$ is a solution to the recurrence relation IFF

$$a_n = \propto_1 (-1)^n + \propto_2 2^n + \propto_3 3^n$$

for some constants $\propto_1$, $\propto_2$, and $\propto_3$

Multiple Distinct Roots, Example II — Cont.

From the initial conditions, it follows that:

$$\begin{aligned} a_0 &= 4 = \propto_1 (-1)^0 + \propto_2 2^0 + \propto_3 3^0 \\ &= \propto_1 + \propto_2 + \propto_3 \end{aligned}$$

$$\begin{aligned} a_1 &= \text{-9} = \propto_1 (-1)^1 + \propto_2 2^1 + \propto_3 3^1 \\ &= -\propto_1 + 2\propto_2 + 3\propto_3 \end{aligned}$$

$$\begin{aligned} a_2 &= \text{-9} = \propto_1 (-1)^2 + \propto_2 2^2 + \propto_3 3^2 \\ &= \propto_1 + 4\propto_2 + 9\propto_3 \end{aligned}$$

Multiple Distinct Roots, Example II — Cont.

Solving: $\propto_1 = 5$, $\propto_2 = 1$, and $\propto_3 = -2$

Therefore, the **solution** to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with:

$$a_n = 5(-1)^n + 2^n - 2(3)^n.$$

Solutions to General Recurrence Relations

The next theorem states the most general result about linear homogeneous recurrence relations with constant coefficients, allowing the characteristic equation to have **multiple** roots.

Key Point: for each root r of the characteristic equation, the general solution has a summand of the form $P(n)r^n$, where $P(n)$ is a polynomial of degree $m - 1$, with m **the multiplicity of this root**.

Theorem 4. Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \ldots - c_k = 0$$

- has t distinct roots, $r_1, r_2, \ldots, r_t$, with
- multiplicities $m_1, m_2, \ldots, m_t$, respectively, so
- $m_i \geq 1$ for $i = 1, 2, \ldots, t$, and
- $m_1 + m_2 + \ldots + m_t = k$.

Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

if and only if

$$\begin{aligned} a_n = & (\propto_{1,0} + \propto_{1,1} n + \ldots + \propto_{1,m_1-1} n^{m_1-1}) r_1^n \\ + & (\propto_{2,0} + \propto_{2,1} n + \ldots + \propto_{2,m_2-1} n^{m_2-1}) r_2^n \\ + & \ldots \\ + & (\propto_{t,0} + \propto_{t,1} n + \ldots + \propto_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

for $n = 0, 1, 2, \ldots,$ where the $\propto_{i,j}$ are constants

for $1 \leq i \leq t$ and $0 \leq j \leq m^i - 1$

Multiple Roots, Example I

If a linear homogeneous recurrence relation has a characteristic equation with roots 2, 2, 2, 5, 5, and 9, then the form of a general solution is:

$$\begin{aligned} a_n = {} & (\propto_{1,0} + \propto_{1,1} n + \propto_{1,2} n^2)2^n \\ & + (\propto_{2,0} + \propto_{2,1} n)5^n \\ & + (\propto_{3,0})9^n \end{aligned}$$

Multiple Roots, Example II

Let: $a_0 = 1$, $a_1 = -2$, $a_2 = -1$, **and**
$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$

We see that $c_1 = -3$, $c_2 = -3$, and $c_3 = -1$

Characteristic Equation: $r^3 + 3r^2 + 3r + 1 = 0$

Since $r^3 + 3r^2 + 3r + 1 = (r+1)^3$, the characteristic equation has a single root, $r = -1$, of multiplicity three.

By Theorem 4., the solutions of this recurrence relation are of the form:

$$a_n = \propto_{1,0} (-1)^n + \propto_{1,1} n(-1)^n + \propto_{1,2} n^2(-1)^n$$

for some constants $\propto_{1,0}$, $\propto_{1,1}$, and $\propto_{1,2}$

Multiple Roots, Example II — Cont.

From the initial conditions, it follows that:

$$\begin{array}{rcrcl}
a_0 & = & 1 & = & \propto_{1,0}(-1)^0 + \propto_{1,1} 0^1(-1)^0 + \propto_{1,2} 0^2(-1)^0 \\
a_1 & = & -2 & = & \propto_{1,0}(-1)^1 + \propto_{1,1} 1^1(-1)^1 + \propto_{1,2} 1^2(-1)^1 \\
a_2 & = & -1 & = & \propto_{1,0}(-1)^2 + \propto_{1,1} 2^1(-1)^2 + \propto_{1,2} 2^2(-1)^2
\end{array}$$

or

$$\begin{array}{rcl}
1 & = & \propto_{1,0} \\
-2 & = & -\propto_{1,0} - \propto_{1,1} - \propto_{1,2} \\
-1 & = & \propto_{1,0} + 2\propto_{1,1} + 4\propto_{1,2}
\end{array}$$

Multiple Roots, Example II — Cont.

Solving these three equations simultaneously yields:

$$\propto_{1,0} \; = \; 1, \qquad \propto_{1,1} \; = \; 3, \qquad \propto_{1,2} \; = \; -2$$

Thus, the unique solution to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with:

$$a_n \; = \; (1 \; + \; 3n \; - \; 2n^2)(-1)^n$$

Multiple Roots, Example III

Let: $a_0 = 1$, $a_1 = 1$, $a_2 = 2$, **and** $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$

We see that $c_1 = 3$, $c_2 = -3$, and $c_3 = 1$

Characteristic Equation: $r^3 - 3r^2 + 3r - 1 = 0$

Since $r^3 - 3r^2 + 3r - 1 = (r-1)^3$, the characteristic equation has a single root, $r = 1$, of multiplicity three.

By Theorem 4., the solutions of this recurrence relation are of the form:

$$a_n = \propto_{1,0} (1)^n + \propto_{1,1} n(1)^n + \propto_{1,2} n^2(1)^n$$

for some constants $\propto_{1,0}$, $\propto_{1,1}$, and $\propto_{1,2}$

Multiple Roots, Example III — Cont.

From the initial conditions, it follows that:

$$\begin{array}{lllllllll} a_0 & = & 1 & = & \alpha_{1,0}\,(1)^0 & + & \alpha_{1,1}\,0^1(1)^0 & + & \alpha_{1,2}\,0^2(1)^0 \\ a_1 & = & 1 & = & \alpha_{1,0}\,(1)^1 & + & \alpha_{1,1}\,1^1(1)^1 & + & \alpha_{1,2}\,1^2(1)^1 \\ a_2 & = & 2 & = & \alpha_{1,0}\,(1)^2 & + & \alpha_{1,1}\,2^1(1)^2 & + & \alpha_{1,2}\,2^2(1)^2 \end{array}$$

or

$$\begin{array}{lllllll} 1 & = & \alpha_{1,0} & & & & \\ 1 & = & \alpha_{1,0} & + & \alpha_{1,1} & + & \alpha_{1,2} \\ 2 & = & \alpha_{1,0} & + & 2\,\alpha_{1,1} & + & 4\,\alpha_{1,2} \end{array}$$

Solving these three equations simultaneously yields:

$$\alpha_{1,0} = 1, \qquad \alpha_{1,1} = -\tfrac{1}{2}, \qquad \alpha_{1,2} = \tfrac{1}{2}$$

Multiple Roots, Example III — Cont.

Thus, the unique solution to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with:

$$\begin{aligned} a_n &= (1 - \tfrac{1}{2}n + \tfrac{1}{2}n^2)(1)^n \\ &= 1 - \tfrac{1}{2}n + \tfrac{1}{2}n^2 \\ &= \frac{2 - n + n^2}{2} \end{aligned}$$

Multiple Roots, Example IV

Let: $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, **and** $a_n = 2a_{n-2} - a_{n-4}$

We see that $c_1 = 0$, $c_2 = 2$, $c_3 = 0$, and $c_4 = -1$

Characteristic Equation: $r^4 - 0r^3 - 2r^2 - 0r + 1 = 0$

or, $r^4 - 2r^2 + 1 = 0$

Since $r^4 - 2r^2 + 1 = (r^2 - 1)^2 = (r-1)^2(r+1)^2$, the characteristic equation has two roots, $r_1 = 1$ and $r_2 = -1$, each of multiplicity two.

Solutions of this recurrence relation are of the form:

$$a_n = (\propto_{1,0} + \propto_{1,1}\, n)(1)^n + (\propto_{2,0} + \propto_{2,1}\, n)(-1)^n$$

for some constants $\propto_{1,0}$, $\propto_{1,1}$, $\propto_{2,0}$, and $\propto_{2,1}$

Multiple Roots, Example IV — Cont.

From the initial conditions, it follows that:

$$\begin{aligned}
a_0 &= 0 = (\propto_{1,0} + \propto_{1,1} 0^1)(1)^0 + (\propto_{2,0} + \propto_{2,1} 0^1)(-1)^0 \\
&= \propto_{1,0} + \propto_{2,0}
\end{aligned}$$

$$\begin{aligned}
a_1 &= 1 = (\propto_{1,0} + \propto_{1,1} 1^1)(1)^1 + (\propto_{2,0} + \propto_{2,1} 1^1)(-1)^1 \\
&= \propto_{1,0} + \propto_{1,1} - \propto_{2,0} - \propto_{2,1}
\end{aligned}$$

$$\begin{aligned}
a_2 &= 2 = (\propto_{1,0} + \propto_{1,1} 2^1)(1)^2 + (\propto_{2,0} + \propto_{2,1} 2^1)(-1)^2 \\
&= \propto_{1,0} + 2\propto_{1,1} + \propto_{2,0} + 2\propto_{2,1}
\end{aligned}$$

$$\begin{aligned}
a_3 &= 3 = (\propto_{1,0} + \propto_{1,1} 3^1)(1)^3 + (\propto_{2,0} + \propto_{2,1} 3^1)(-1)^3 \\
&= \propto_{1,0} + 3\propto_{1,1} - \propto_{2,0} - 3\propto_{2,1}
\end{aligned}$$

Solving these three equations simultaneously yields:

$$\propto_{1,0} = \propto_{2,0} = \propto_{2,1} = 0 \quad \text{and} \quad \propto_{1,1} = 1$$

Multiple Roots, Example IV — Cont.

Thus, the unique solution to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with:

$$
\begin{aligned}
a_n &= (0 + 1n)1^n + (0 + 0n)(-1)^n \\
&= n
\end{aligned}
$$