Mat 3770
Steiner Trees

Spring 2014

Topics

- Background, Definitions, and Prior Theorems
- Initial Algorithms
- Theoretical Work
- Improved Performance Algorithms
- Conclusions

Motivation for Studying Steiner Trees

Applications

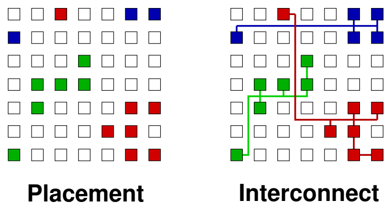
- Communications networks
- Mechanical & Electrical systems in buildings and along streets
- Wire layout in VLSI chip design

VLSI Chip Design

Given a collection of cells and a collection of nets, find a way to position the cells (*placement*) and run the wires for net connections (*routing*) so the wires are short with as few vias as possible, and the whole layout uses a minimum amount of area.

Gate Arrays

- Custom vs Semi–custom chip design
- A two dimensional array of replicated transistors fabricated just short of the interconnection phase, allowing customized connections to define the overall circuits for semi–custom design chips.
- The interconnections are implemented on a *rectangular grid* in the channels between the cells.



Fermat's Problem

- In the early 1600's, Pierre Fermat posed the problem:
  Given a triangle, find the point in the plane such that the sum of the distances to the vertices is minimized.
- Evangelista Torricelli solved this problem in 1659:
  If all angles are less than 120^o, then P is the point from which each side of the triangle subtends an angle of 120^o, else it is the vertex of largest measure.

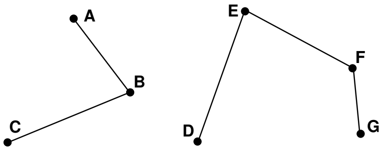
Steiner's Problem

In the early 1800's, Jacob Steiner formalized the problem mathematically and generalized it to *n* points:

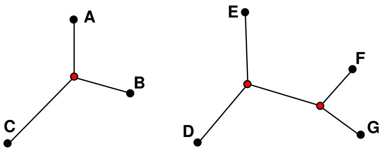
Given n points in a plane, find a connected system of straight line segments of shortest total length such that any two of the given points can be joined by a path consisting of segments of the system.

Spanning Trees

Related Problem: Minimal Spanning Trees

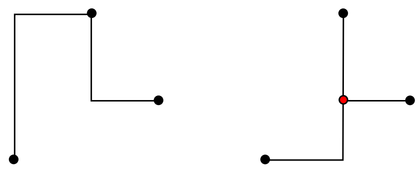


Steiner Minimal Trees: *shorten trees by adding points*



Shortest Rectilinear Steiner Spanning Tree Problem

Connecting a set of points in the plane with a connected collection of vertical and horizontal lines with minimal overall length.

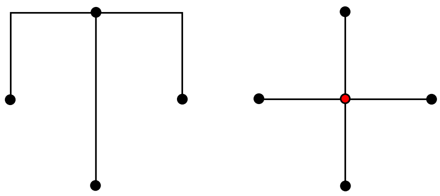


Rectilinear Minimal Spanning Tree

Shortest Rectilinear Steiner Spanning Tree
flip connections and take out overlaps

Upper and Lower Bounds

Theorem. (*Hwang*) An SRSST over a set of points is no smaller in length than two thirds the length of the RMST over the same set of points.

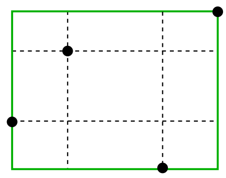


RMST **SRSST**

This gives us nice upper (Length(RMST)) and lower ($\frac{2}{3}$ Length(RMST)) bounds on the length of Steiner trees.

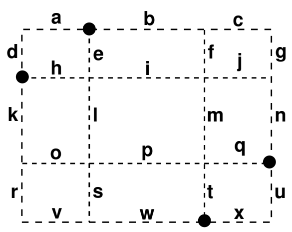
Grid & Enclosing Rectangle

- **Theorem.** (*Hannan*) An SRSST over a set of points exists on the grid induced by the points.



- Note: The shortest path between two points is not unique. Thus a solution to the SRSST Problem in general is not unique.
- The **Enclosing Rectangle** is the smallest rectangle containing the point set. Any SRSST must be at least half the perimeter of the enclosing rectangle. (Providing another lower bound.)

Grid Segments



Points and Their Induced Grid

Number of Segments

(Horizontal & Vertical) $\times$ n nodes $\times$ $(n-1)$ edges

4 points:	$2 \times 4 \times 3$	$=$	24 grid segments
5 points:	$2 \times 5 \times 4$	$=$	40 grid segments
6 points:	$2 \times 6 \times 5$	$=$	60 grid segments

NP–Complete Problem

Theorem. (*Garey and Johnson*) The problem of determining the minimum length of an optimal rectilinear Steiner tree for a set of points in the plane is NP–Complete.

This implies it is probable that finding an optimal solution will take worse than polynomial time.

That may be alright for small problems, but heuristic algorithms are still needed.

Heuristic Algorithms

Non–optimal, but usually close
& have performance guarantees

Research Sequence

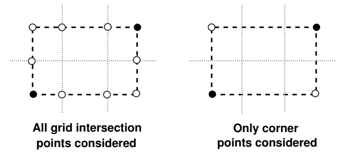
- Extended Kruskal
- Naïve Branch and Bound (optimal)
- Structural Theorems
- Pretaxial & Epitaxial Branch and Bound (optimal)
- MST–based Direction Assignment
- Simulated Annealing
- Stochastic Evolution

Extended Kruskal

- Based on Kruskal's MST algorithm:

```
Place all points in individual subtrees
Repeat
   Connect closest subtrees
Until a single tree is formed
```

- Two versions with different connection schemes:



- **Theorem.** The EK algorithm is correct and produces an RSST no larger than the RMST.

  Guarantees result is no worse than 150% of optimal

  .

Naïve Branch and Bound

Idea: look at all combinations of grid edges.

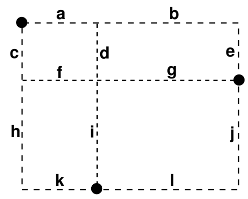
General Branch and Bound Method

Examine all feasible solutions
in an orderly manner:

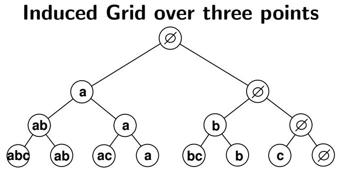
Organize the search space as a tree

Search the tree using a depth first approach
Using **lower** *and* **upper** *bounds*
to decrease the search

Grid and Search Tree



Induced Grid over three points



. . . etc . . .

Naïve Search Tree

Pruning the Search Tree

- A small search space is needed to **feasibly** complete the tree traversal, so we **prune** the tree:
  - **lower bound**: larger of $\frac{2}{3}$ RMST or $\frac{1}{2}$ perimeter
  - **upper bound**: RMST or result of heuristic
- If collection of grid edges is
  - too large — discard
  - too short — discard
  - if they form a tree and it's shorter than current known tree, keep it
- Unfortunately, there were still too many combinations to check

Implementation

- All combinations of grid edges considered

  Search Space: $O(2^{2n(n-1)})$
  3 points yield 2^{12} or 4096 combinations

- Implementation: essentially used a binary odometer where $1 = IN$ and $0 = OUT$.
- Order the grid edges:

l	k	j	i	h	g	f	e	d	c	b	a
0	0	0	0	0	0	0	1	1	0	1	0

The 1's at *e*, *d*, and *b* represent a forest containing those edges and no others.

- Determining whether the edges formed a tree (i.e., if they are connected) was **very** time consuming and the Search Space was too large.

Structural Theorems — Definitions

Idea: reduce the number of configurations must consider

- A **line** is a connected collection of one or more grid segments, all with the same orientation (either horizontal or vertical).
- A **via** occurs the intersection of a horizontal and a vertical grid segment.
- A **non–repetitive** set of points contains no duplicate x or y coordinates. I.e., all x and y coordinates are unique.

  A point set can be perturbed to meet this requirement.

Structural Theorems — Theorems

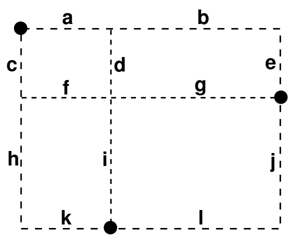
- **Theorem.** There exists an SRSST over a set of n points which has at most n lines.
- **Theorem.** If an SRSST has n lines, it has $n-1$ vias (the fewest possible).
- **Theorem.** Given a non–repetitive point set with at most one corner point, there exists an SRSST over the set which contains an inner line from an arbitrary outer point.
- **Corollary.** If a non–repetitive point set contains a corner point, then there is an SRSST over the set of points which contains an arbitrarily chosen outer line adjacent to the corner point.

More Branching and Bounding

The previous (and other) theoretical work was used to develop two new Branch and Bound algorithms.

- **Pretaxial** — grow a collection of lines, one through each point, then check to see if it forms a tree
- **Epitaxial** — grow a **connected** set of lines, one through each point, then check to see if it's shortest

Pretaxial Branch and Bound

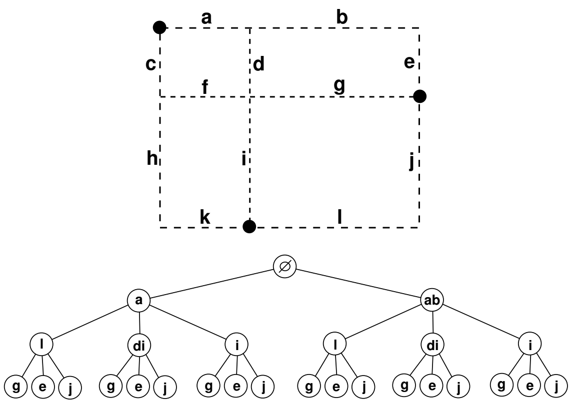


Induced Grid over Point Set

Line Table		
top	bottom	right
a	di	**fg**
ab	i	g
c	**k**	e
ch	l	j
	kl	**ej**

Lines listed in **red** have been eliminated

Pretaxial Search Tree



Tuning the Algorithm

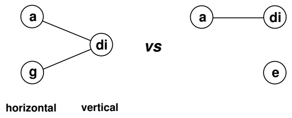
- To aid in pruning, we can order both the points within the table as well as the lines for each point.
- As we traverse the tree, we backtrack (prune) whenever our collection of lines:
  - contains a line through each point (*reached leaf*)
  - contains a cycle (*too long*)
  - is longer than the best tree found so far (*discard immediately*)
  - will never span the points (*too short*)
- **Theorem.** The *Pretaxial* Branch and Bound search tree contains an SRSST over the set of points.

The Algorithm

```
Procedure ExamineTree(PointSet,
        Edges, lower, Best)
Initialize(T)
status = continue
Repeat
   if not Feasible(T, E)
      status = backtrack
   else if Complete(T, P)
      if Better(T, Best)
         Replace(Best, T)
      status = backtrack
   if status = backtrack
      if UnexpandedNode(T, E)
         Backtrack(T, E)
         AddEdge(T, E)
         status = continue
      else
         status = stop
   else
      if LastConfig(T, E)
         status = stop
      else
         if AtLeaf(T, E)
            Backtrack(T, E)
            AddEdge(T, E)
Until status = stop
```

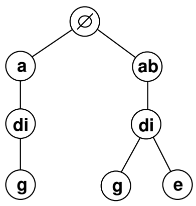

Determining Whether Tree Is Connected

- How to determine whether the forest formed by edges from the table is **connected** (and thus a tree)?
- Since only horizontal and vertical lines can connect, these connections form a **Bipartite Graph**.
- Thus, vertices can be divided into two sets such that no two vertices in the same set are connected by an edge.
- Apply a depth–first search, marking vertices as they are visited. If all vertices are marked in that process, the tree is connected.



Epitaxial Approach — “Grow your own tree”

- Ensure new edges to be added to the tree intersect an edge already in the tree
- Need an **intersecting lines** table
- Form a **pool** of candidate lines from which to choose next line
- Still need only one line through each point



Only 3 feasible configurations result

Notes

When reducing the Search Space, be careful **not to eliminate** all optimal solutions: you must be able to show or prove at least one optimal solution still exists and will be found.

A possible heuristic: do a study to see **where in the search tree** Best is found. For example, an optimal solution for this problem was found in the first three branches of the root, so perhaps only look there. (The edges were ordered smallest to largest at each level.)

It may also be possible to **parallelize** the search — hand off subtrees to various processors to search and report back.

Heuristic Algorithms

The MST–Based Depth First Direction Assignment Algorithm, based on the Structural Theorems

Idea: represent a spanning tree as **connection pairs** (of vertices or points), then force each point to have one and only one line associated with it (either horizontal or vertical), resulting in **intersection pairs**.

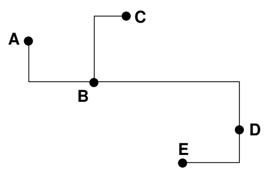
Importance: A new data structure to represent a special class of Steiner Trees.

RMST Representation

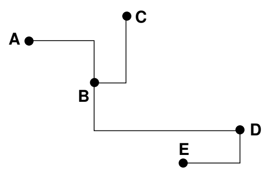
Given n vertices, there will be $n - 1$ connection pairs

Graphical versions are not unique.

MST list: `<A, B>`
`<B, C>`
`<B, D>`
`<D, E>`



Version 1

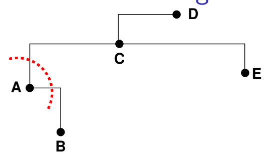


Version 2

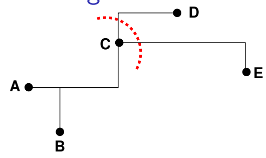
We want to associate one line per point, so assign a direction to each point, either `Horizontal` or `Vertical`.

Once we have obtained a tree, we can form another tree from it by flipping the direction of the line associated with each point.

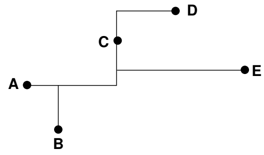
Depth–First Processing: Direction Assignment



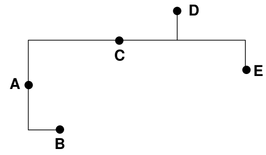
Beginning Sweep



Continuing Sweep



Tree 1



Tree 2

RSST Representation

- Intersection Pairs define a unique tree.
- Given a Minimum Spanning Tree, it is easy to process:
  - build a Pairs Table, detailing which points connect to each point
  - Use Breadth–First processing (a ripple effect)

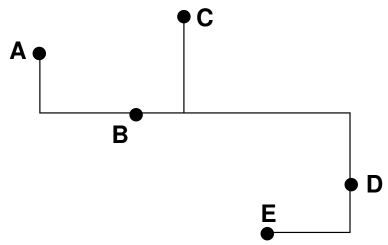
Tree 1

Intersection Pairs

— <H, V> —

<B, A>
<B, C>
<B, D>
<E, D>

Graphical Representation



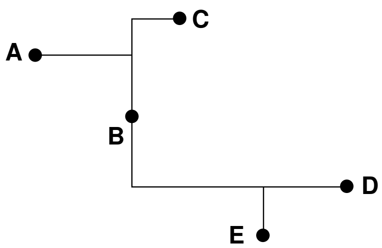
Tree 2 — Built by Swapping Orientations

Intersection Pairs

— <H, V> —

<A, B>
<C, B>
<D, B>
<D, E>

Graphical Representation



More Theorems

Theorem. Using the set of direct connections denoted by an RMST over a set of points, we can always derive from them exactly two Rectilinear Steiner Spanning Trees containing one line per point over the same set of points.

Theorem. The MST–Based Depth First Direction Assignment Algorithm is correct and produces a tree which is no longer than 1.5 times optimal.

Theorem. Any Rectilinear Steiner Tree of n lines over n points may be represented by this data structure.

Simulated Annealing

Idea: Using the Intersection pairs data structure, apply the Simulated Annealing algorithm (Kirkpatrick, Gelat & Vecchio) to the RSST Problem.

- Simulated Annealing is based on a strong analogy between the **physical** annealing process of solids and the process of solving large combinatorial optimization problems.
- Two phases of annealing:
  - **Melting** the solid so particles arrange themselves randomly — must have sufficiently high initial temperature.
  - Careful **temperature reduction** until particles arrange themselves in the ground state of the solid where the energy of the system is minimal

The Model

Two criteria for obtaining the ground state:

- Sufficiently **high** maximum temperature
- Sufficiently **slow** cooling schedule

To model the annealing process we need:

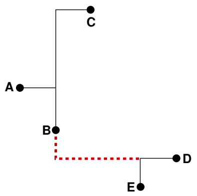
1. Definition of a **configuration**, and an **initial feasible solution** described as such a configuration: **Intersection Pairs**.
2. A method for generating a **neighboring configuration**, which must also represent a feasible solution: **Randomly choose a pair to replace**.
3. A method for evaluating the **objective function** of the problem for any configuration: **length of tree**.

Configuration & Random Neighbor

- Randomly choose pair #3 to replace
- Always results in two subtrees
- Neighbor is longer, but may still be accepted

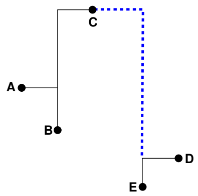
(a) Initial

```
<A, B>
<C, B>
<D, B>
<D, E>
```



(b) Neighboring

```
<A, B>
<C, B>
<C, E>
<D, E>
```



Stochastic Evolution

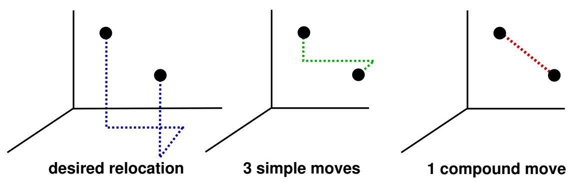
Idea: Using the Intersection pairs data structure, apply the Stochastic Evolution algorithm (Saab, Rao, '91) to the RSST Problem.

Stochastic Evolution is an **adaptive heuristic combinatorial optimization** algorithm based loosely on biological evolution.

Similar to Simulated Annealing

- Stochastic Evolution uses the same configurations as Simulated Annealing
- Neighboring configurations are derived by **compound** moves instead of the simple moves of Simulated Annealing
- Stochastic Evolution keeps track of the **best found so far**, whereas Simulated Annealing converges to a solution
- Doesn't require such careful tuning of control parameters; used less time and was more effective

Simple vs Compound Moves



SE Algorithm

```
SteinerSE(Config, Best, InitialMaxOver, MaxCount)

// PREConditions:
  Config          :an initial feasible configuration
                     given as intersection pairs
  InitialMaxOver  :initial maximum negative gain
  MaxCount        :approximate number of iterations
                     before improvement is found

// POSTCondition:
  Best            :configuration of least cost
                     found by algorithm

// Initializations
  Best = Config
  Count = 0
  MaxOver = InitialMaxOver
```


SE Algorithm, Continued

```
// Iteratively search for better solution
REPEAT

   // Create new solution
   PreCost = Length(Config)
   Perturb(Config, MaxOver)
   PostCost = Length(Config)
   Update(MaxOver, PreCost, PostCost, InitialMaxOver)

   if (PostCost < Length(Best)
      // if improved, save best and reward counter
      Best = Config
      Count -= MaxCount
   else
      Count++

UNTIL Count > MaxCount
```


Theorem. The SteinerSE Algorithm produces a tree which is no longer than 1.5 times optimal.

Table 1. Empirical Results SteinerSE Statistics

Set Size	MST cost	Time	% Improvement
10	50	1.5	10.8
15	70	6	9.9
20	125	21	11.1
25	125	30	10.9
30	120	39	12.3
35	120	57	10.9
50	70	91	11.7
100	40	550	11.1
		Overall Ave:	11.1

Table 2. Comparison of Results to Best Found

Algorithm	Best of 60	% of Best	% Off
EK1	21	35.0	2.073
EK2	25	41.7	2.205
EK1 & EK2	25	41.7	2.124
MDFDA	17	28.3	4.039
SE	55	**91.7**	**0.864**

Table 3. Comparison of Reported Results

Researchers	Method	% Improvement
Lee et al	Prim	9
Hwang	Prim	9
Bern et al	Kruskal	9
Richards	line–sweep	4
Lewis et al	line–sweep	8.4
Smith et al	geometric	8
Ho et al	edge–flip	9
Kahng et al	1–Steiner	10.9
SteinerSe	Combinatorial Optim.	**11.1**

Conclusions

When time is not limited:

- For 12 or fewer points: *Epitaxial Branch & Bound*
- For up to 20 points: *Stochastic Evolution*
- For 35^+ points: *ExtendedKruskal*

Otherwise:

- *ExtendedKruskal*
- *DirectionAssignment*, if nearly linear time is required

All of our algorithms have a performance guarantee