Mat 3770

Week 2

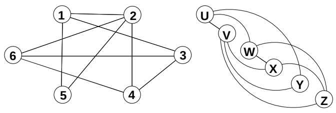
Spring 2014

Week 2 — Student Responsibilities

- Reading: Isomorphism, Edge Counting, Planar Graphs
- Homework from Tucker & Rosen
- Attendence **Slushily** Encouraged

Section 1.2. Isomorphism

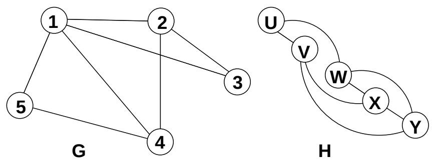
- Two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ are said to be **isomorphic** if there exists a bijection $f : V_G \to V_H \ni$
  $$< u, w > \in E_G \text{ IFF } < f(u), f(w) > \in E_H$$
- I.e., we can relabel the vertices of G to be vertices of H, maintaining the corresponding edges in G and H; pairs are adjacent in G IFF pairs are adjacent in H



- The mapping from V_G to V_H given by
  ```
  f(1) = u,  f(2) = v,  f(3) = w,
  f(4) = x,  f(5) = y,  f(6) = z
  ```
  is the requisite bijection.

Isomorphism

- These two graphs are not isomorphic since $\deg(1) = 4$, and no vertex in graph H has degree 4.



- Note: degrees are preserved under isomorphism

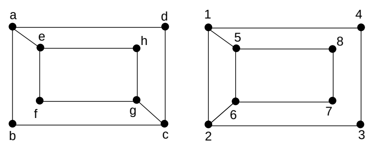
Isomorphic Graphs

- Same number of vertices
- Same number of edges
- same number of vertices with a given degree
- corresponding edges are maintained between vertices of same degree as pre–image.

A graph $H = (V_H, E_H)$ is a **subgraph** of $G = (V_G, E_G)$ if $V_H \subseteq V_G$ and $E_H \subseteq E_G$.

Isomorphic Subgraphs

- If we cannot find isomorphic subgraphs, then the graphs are not isomorphic.




```
deg 2:  b,d,f,h        3,4,8,7
deg 3:  a,c,e,g        1,2,5,6
```

- Subgraphs containing these (deg 2) vertices must be isomorphic.
- No edges between `b,d,f,` or `h` (within same set), while edges `<3,4>` and `<7,8>` exist. Therefore the two graphs are **not** isomorphic.

Recall

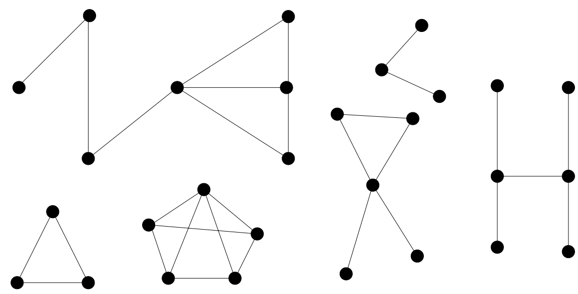
- **Matching**: Set of edges, none of which share the same endpoint
- **Maximum matching**: Maximum cardinality of all matchings
- **Edge Cover**: Set of vertices which contains at least one endpoint of all edges in graph

- **Independent Set**: Set of vertices no two of which share an edge
- **Maximal Independent Set**: Cannot add any other vertex in the graph and remain independent (i.e., every vertex not in the set is adjacent to some vertex in the set)
- **Maximum Independent Set**: Maximum cardinality of all Independent sets
- **Theorem**: Given a graph G=(V, E), if S⊂V is independent, then v−S is an edge cover and vice versa

Applications

Connected Components

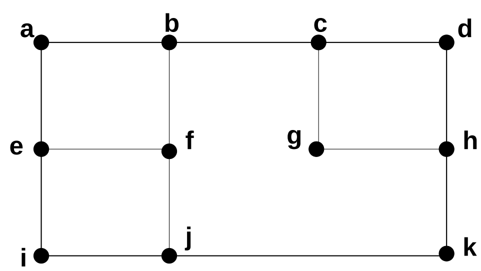
- One of the incentives for developing the Internet was the threat of war and the fear of having communications between various installations in the United States severed.
- Given a graph, can we determine if there is a **critical** edge, one whose removal disconnects the graph?



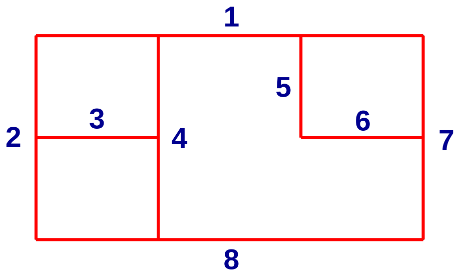
Applications

Edge Cover

The Manhattan Police Department (MPD) knows several heads of organized crime are meeting in a particular area of the city and want to keep the streets there under surveillance. Unfortunately, owing to budget constraints, they need to use the fewest officers possible.



How can we determine on which corners to place officers to maximize their usefulness (the number of adjacent blocks they can observe) while minimizing the number of officers?



We can replace the original 14 graph edges with the 8 contiguous line segments which they comprise, forming another, slightly different graph to model the problem.

Applications

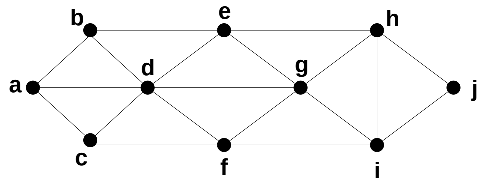
Scheduling Problems

Suppose we allowed students to sign up for courses, then scheduled the courses so the total number of hours needed is minimized, and no two classes which share students meet at the same time.

This can be modeled with a graph where each class is a vertex, and an edge between two vertices means they share at least one student.

A set of courses can all meet at the same time if there are no edges between any of them, i.e., they form an independent set. Thus, we need to find the minimum number of independent sets that collectively include all vertices.

Consider also that if we find a maximum independent set, we'll have a minimum edge cover, and vice versa. Thus, finding a **maximum independent set** is **equivalent** to finding a **minimum edge cover**.



Applications

Not everyone has a telephone, nor even a permanent address. Suppose we needed to get word out "on the street" about a new program to help the homeless.

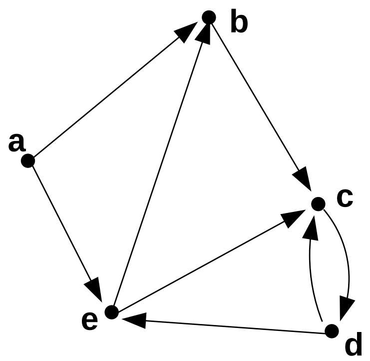
We can use a graph to represent this problem. Vertices are people, and the directed edges between them represent "is able to contact."

Because we have so little time, we want to find a minimal subset of people who can spread the word to the whole group—either directly or by word of mouth.

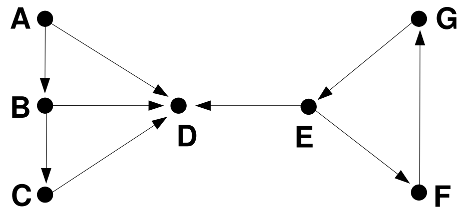
We want a a **vertex basis — a minimal set of vertices with directed paths to all other vertices**.

We can build a **directed–path graph** for the original graph with the same vertex set and with a directed edge $< p_i, p_j >$ added if there is a directed path from p_i to p_j in the original graph.

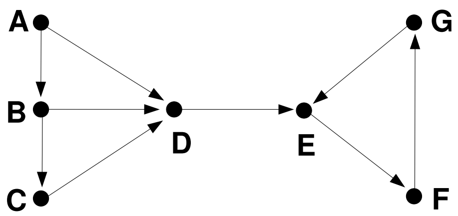
Best Way to Spread the News Quickly?



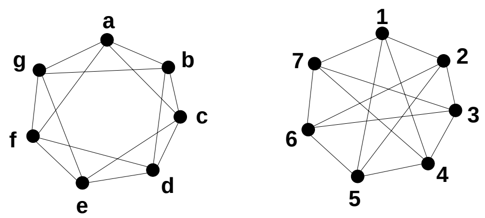
Find a Vertex Basis



Find a Vertex Basis



Are these graphs isomorphic?



Note: the **complements** of isomorphic graphs are isomorphic.

Exercise : Find non–isomorphic directed graphs with three vertices:

Section 1.3. Edge Counting

- **Theorem 1**. In any graph, the sum of the degrees of all vertices is equal to twice the number of edges.

  Q1. Given G = (V, E), how many vertices are there in V if there are 15 edges in E, and all vertices have degree 3?

- **Corollary**. In any graph, the number of vertices of odd degree is even.

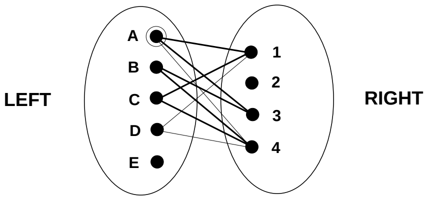
  Q2. How many edges are in K_n, a complete graph over n vertices?

Mountain Range Problem

Example 4, which begins on page 29, is worded a bit obscurely. Part of the homework for this section is to “rephrase” the problem in simple English.

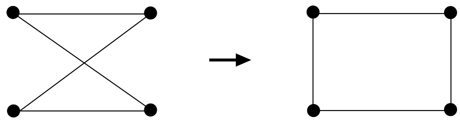
Bipartite Graphs and Circuits

- (**Recall**) The **length** of a circuit or path is the number of edges in it.
- **Theorem 2**. A graph G is **bipartite** IFF every circuit in G has even length.



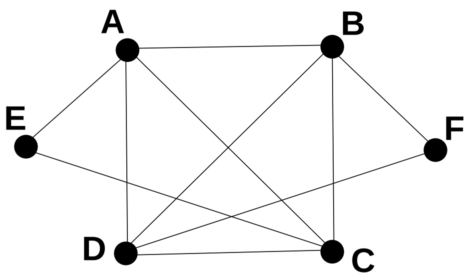
Section 1.4 Planar Graphs

- A graph is **planar** if it can be drawn on a plane with no edges crossing. For example:

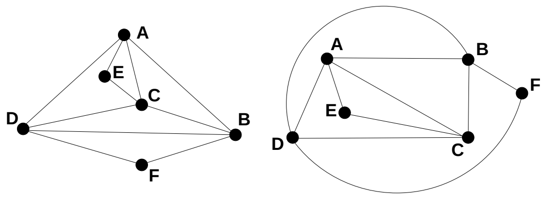


The graph on the left can be redrawn with no edges crossing; therefore, it is a planar graph.

Figure 1.21a, Pg 35 – **Not so Obvious**!



But It **Is** Planar



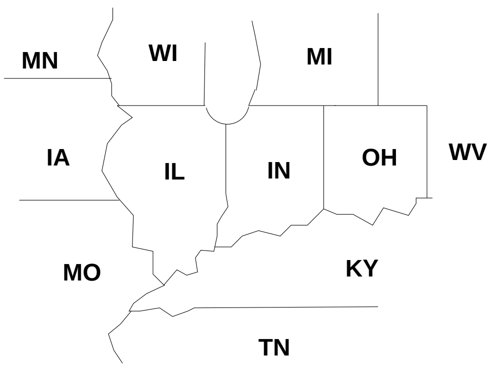
Computer–Related Uses

- Wiring diagrams for each layer on a computer chip — if wires cross, they share electricity (not good)
- Robotic motion planning (2–D) — plot obstacles, determine path
- Map Coloring

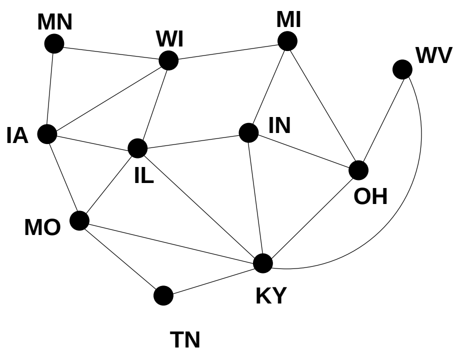
  How many colors are needed to color counties (states, etc.) on some map in order that adjacent counties all have different colors?

  A map can be modeled by a planar graph.
  USA map: states are vertices; edges indicate "share a border"

US State Map



Modeling Maps with Graphs



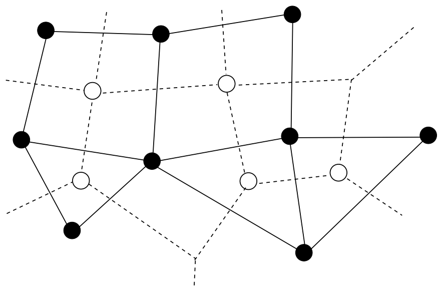
Maps

- **Map**: a planar graph with edges as borders and vertices where borders meet.
- The **map coloring** problem may be stated as follows:

  `What is the minimum number of colors needed so adjacent vertices have different colors?`
- **Dual Map**: make a vertex for each planar face; then an edge between vertices corresponds to adjacent faces.

  An "extra" vertex is added for the unbounded region surrounding the map:

Example



Map Coloring Solution

- **Famous conjecture**: no planar graph needs more than four colors.
- This conjecture was "proved" in 1976 by 2 guys at U of I using a computer for an "exhaustive" proof (i.e., they considered all possible configurations).
- Many purists didn't like their proof because it wasn't **elegant**!

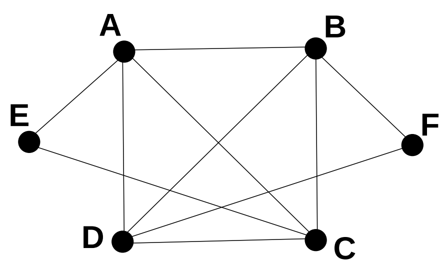
The Circle–Chord Method

Problem:
Given a general graph, draw it as a planar graph (if possible)

- The **Circle–Chord Method** for drawing planar graphs:
  1. Find a circuit which contains all (or most) of the vertices in the graph
  2. Draw this circuit as a large circle
  3. The rest of the edges (chords) must be either inside or outside the circle
  4. Choose one chord and draw it, say inside the circle
  5. This may force some chords inside, some outside
  6. Keep adding chords until all are in the graph **or** until adding a chord, either inside or out, will cross an existing edge (in which case the graph is non–planar).

Example

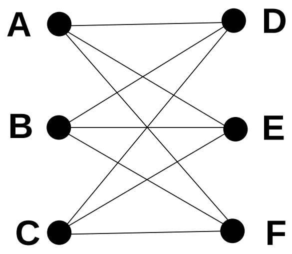
Find a planar depiction of this graph using the circle–chord method:



Begin by finding a circuit that contains all the vertices.

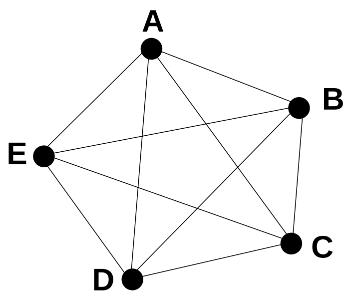
$K_{3,3}$

$K_{3,3}$: a complete bipartite graph with two sets of three vertices, with each vertex in one set adjacent to the three vertices in the other set. Is it planar?



K_5

K_5: a complete graph of 5 vertices. Is it planar?



More on Planarity

- **Subdividing** a graph: adding vertices to the middle of zero or more edges:

  Note: subdividing a non–planar graph cannot make it planar.

- $K_{3,3}$ **Configuration**: a (sub)graph obtained by subdividing a $K_{3,3}$.
- K_5 **Configuration**: a (sub)graph obtained by subdividing a K_5.
- **Theorem 1**. A graph is planar *IFF* it does **not** contain a subgraph that is a K_5 or a $K_{3,3}$ configuration.

More Theoretical Results

- **Theorem 2 (Euler's Formula)**. If G is a connected planar graph with $|V| = \text{v}$ and $|E| = \text{e}$, then a planar depiction of G will always have $\text{r} = \text{e} - \text{v} + 2$ regions (areas); r is also known as the number of faces.

- **Corollary**. If G is a connected planar graph with $e > 1$, then $e \leq 3v - 6$.

  note: This is not an IFF statement. If $e \not\leq 3v - 6$, the graph is non–planar; but if $e \leq 3v - 6$, and the graph is connected, there is not enough information to determine if it is planar.

Children, Goats, and Rabbits

- Page 53, #16: Suppose there are three farms each with a child (C), a goat (G), and a rabbit (R).

  1. The male child on the farm with goat G_a and the male child on the farm with rabbit R_c are competing for the attention of the female child C_b on the third farm.
  2. Goat G_C and rabbit R_b are not on the same farm
  3. The boy on the farm with rabbit R_a is not C_a