Week 5 — Student Responsibilities

- Reading: Edge Counting, Planarity (See Syllabus schedule)
- Hwk from Tucker – 2.4
- Hwk from Rosen – 9.8
- Attendance *Sprightfully* Encouraged

What is the Chromatic Number of this Graph?

![Graph](image)

**Graph Coloring Applications—VLSI Chip Design**

**VLSI**: Very Large Scale Integrated Chip Design—the “brains” of a computer

**Gates**: logical sub-circuits in a computer chip which are composed of electronic switches

Possibilities in chip manufacturing:
- Most expensive: Custom fabricated chips
- Medium expense: Semi–custom chips
- Least expensive: “Off–the–Shelf” chips

**Semi–Custom Design**

Using Pre–fabricated Chips
- fabricated up to the inter-connection phase
- reduces overall cost of manufacturing chips
- example: Programmable Logic Arrays (PLA)
  - gates are laid out in rows \( G_i, G_2, \ldots, G_n \) with specified connections between certain pairs, \( G_i \) and \( G_j \), given as \( < i, j > \)
  - connections are laid out in parallel tracks (columns)
  - no connections may overlap, not even at an endpoint
  - we want to minimize the number of tracks required
A Programmable Logic Array Example

Suppose we want the following Gate connections

\[
<1, 3> <1, 3> <2, 3>
<2, 4> <2, 5> <4, 5>
\]

The layout below “realizes” these connections using 5 tracks

\[
\begin{array}{cccc}
G_1 & 1 & 2 & 3 & 4 & 5 \\
G_2 & & & & & \\
G_3 & & & & & \\
G_4 & & & & & \\
G_5 & & & & & \\
\end{array}
\]

Minimizing the Layout

If we have the time and money, we can re-arrange rows and improve the routing phase. Notice that gates in rows 1 & 3 want to be together, as do gates in rows 2 & 3, and in 4 & 5

We get very good improvement: from 5 tracks down to just 3.

How is this modeled in a program? With a graph. We can use a force-directed algorithm, which acts like springs attached to the rows so those with many connections are more attracted than those with fewer.

Interval Graph

- **Interval Graph**: a graph \( G \) with a one-to-one correspondence between its vertices and a collection of intervals on the line such that two vertices of \( G \) are adjacent when the corresponding intervals overlap.

  Example applications: competition graph (used in ecology; species compete for survival), VLSI routing problems (PLA folding).

Given the VLSI design problem of connecting the rows of gates:

\[
(1, 3) (1, 3) (2, 3)
(2, 4) (2, 5) (4, 5)
\]

we can model the problem of determining the minimum number of tracks by finding the Chromatic Number of a related interval graph.

Consider: \( K_5 \) Subgraph

- A complete graph, \( K_5 \), requires 5 colors (and we cannot color it in fewer colors).

  Thus, we need at least 5 tracks.
Another Example

▶ The largest complete graph in the figure below is a triangle, therefore it requires 3 colors (and we cannot color it in 2 colors)
▶ Thus we need only 3 tracks when the rows are rearranged.

Example of a Triangulated Polygon

Triangulation Theorem

Theorem 1. The vertices in a triangulation of a polygon can be 3–colored.

Proof is by induction on $n$, the number of edges in the polygon.

BC. Let $n = 3$.
Then the polygon is a triangle, and clearly can be 3–colored.

IH. (Strong induction) Assume any triangulated polygon with $4 \leq k < n$ boundary edges can be 3–colored for some arbitrary $n \geq 4$.

IS. Show a triangulated polygon, $T$, with $n$ boundary edges can be 3–colored.
▶ Pick some chord edge $e = <v_i, v_j>$, which must exist since $T$ has been triangulated.
▶ Since all chord edges connect vertices of the polygon, the chord edge $e$ splits $T$ into two smaller triangulated polygons, each of which can be 3–colored by the IH.
▶ In each coloring, $v_i$ will have some color, and $v_j$ will have some other color.
▶ Then the two subgraphs can be combined to yield a 3–coloring of the original polygon since, if need be, the coloring of one of the smaller polygons can be modified. Note: this 3–coloring is unique.

Application of the Theorem

▶ Art Gallery Problem: What are the fewest number of guards needed to watch paintings along the $n$ walls of an art gallery?
▶ Guards must have direct line–of–sight to every point on the walls.
▶ A guard at a corner is assumed to be able to see the two walls that end at that corner, and the wall directly opposite the corner, if there is one.
Fisk’s Corollary

**Corollary:** the Art Gallery Problem with \( n \) walls requires at most \( \left\lfloor \frac{n}{3} \right\rfloor \) guards (where \( \lfloor \rfloor \) is the floor function.)

**Proof:** Let the \( n \) walls form a polygon \( P \) with triangulation \( T \). 3-color \( T \) and note each triangle will have a corner of each color. Pick one color, \( c \), and place a guard at each corner colored \( c \) (1 in each triangle). Hence the sides (and thus all walls) of every triangle will be watched.

A polygon with \( n \) walls has \( n \) corners. If there are \( n \) corners and 3 colors, some color is used on \( \left\lfloor \frac{n}{3} \right\rfloor \) or fewer corners.

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Other Coloring Theorems

**Notes:**
- If a graph is bipartite, it is 2–colorable (and vice–versa)
- A graph is 2–colorable IFF all circuits have even length (this doesn’t require the graph to be connected)
- Let \( \chi(G) \) denote the chromatic number of \( G \).

**Theorem 2.** If the graph \( G \) is not an odd circuit or a complete graph, then \( \chi(G) \leq d \) where \( d \) is the maximum degree of a vertex in \( G \). (This gives a usually poor upper bound on \( \chi(G) \))

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Theorem 3. For any positive integer \( k \), there exists a triangle–free graph \( G \) with \( \chi(G) = k \)

Rather than color vertices, we can color edges so that all edges incident to the same vertex must have different colors.

**Theorem 4 (Vizing’s Theorem).** If the maximum degree of a vertex in a graph \( G \) is \( d \), then the edge chromatic number of \( G \) is either \( d \) or \( d + 1 \).

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Theorem 5. Every planar graph can be 5–colored

**Note:** in Tucker, Section 1.4, exercise 16, the reader was asked to prove: Any connected planar graph has a vertex of degree at most 5.

**Theorem 5 Proof — by induction on the number of vertices.**

- **BC.** Let \( 1 \leq n \leq 5 \). Trivially, any such \( n \) vertex graph can be 5–colored.
- **IH.** Assume for some arbitrary \( n \geq 1 \), that connected planar graphs with \( n - 1 \) vertices can be 5–colored.
- **IS.** Show a graph with \( n \) vertices can be 5–colored.

By the note, \( G \) must have a vertex, \( x \), of degree at most 5. Delete \( x \) from \( G \) to obtain a graph, \( G' \), with \( n - 1 \) vertices. By the IH, \( G' \) is 5–colorable.

Now, reconnect \( x \) to the graph and try to properly color \( x \).

If \( x \) has degree \( \leq 4 \), then simply assign a color to \( x \) which is different from any of its neighbors. The same coloring works if the degree of \( x \) is 5 and 2 or more of its neighbors has the same color.

There remains the case of how to color \( x \) if all 5 neighbors have different colors.

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By letting the adjacent vertices \( A, B, C, D, \) and \( E \), imposing a clock–wise ordering around \( X \) in a planar depiction of \( G \).

- Let the colors 1—5 be assigned to vertices \( A—E \) in order.
- Consider vertex \( A \), colored 1, and vertex \( C \), colored 3.
Case 1.

If there is no path between them (other than through A), A may be recolored with color 3, all vertices adjacent to A which are 3 can be assigned 1, and so on.

This re-coloring will not affect C since there is no path from A to C, and furthermore, will only affect vertices reachable from A which are colored 1 or 3.

After the re-coloring, X may be colored 1.

Case 2.

There exists a path from A to C. This path either encompasses B, or it encompasses D, but not both, since G is planar.

Thus there can be no path between B and D (other than through A), so the same type of re-coloring may be applied to B (color 2), using D’s color (4).

Thus allowing X to be colored with 2.

Hence, every planar graph can be 5–colored.

Tucker, Chapter 2 Overview

- Section 2.1 Euler cycles — cycles that traverse every edge exactly once. Determine existence with Euler’s Theorem.
- Section 2.2 Hamilton circuits — circuits that visit every vertex exactly once. Determine existence by a laborious systematic search to try all possible ways of constructing a HC.
- Section 2.3 Graph coloring & some applications.
- Section 2.4 Graph coloring theory.