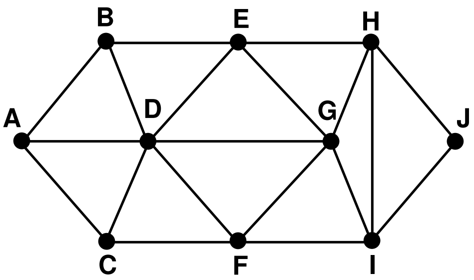
Mat 3770
Week 5

Spring 2014

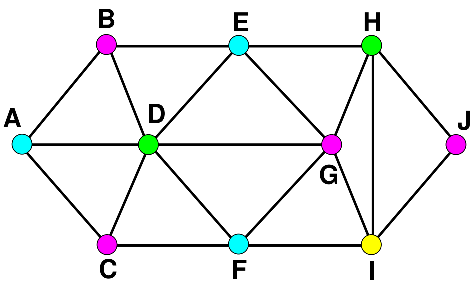
Week 5 — Student Responsibilities

- Reading: Edge Counting, Planarity
  (See Syllabus schedule)
- Hwk from Tucker – 2.4
- Hwk from Rosen – 9.8
- Attendance **Sprightfully** Encouraged

What is the Chromatic Number of this Graph?



An Example Coloring



Graph Coloring Applications—VLSI Chip Design

VLSI: Very Large Scale Integrated Chip Design—the “brains” of a computer

Gates: logical sub-circuits in a computer chip which are composed of electronic switches

Possibilities in chip manufacturing:

- Most expensive: **Custom fabricated** chips
- Medium expense: **Semi–custom** chips
- Least expensive: “**Off–the–Shelf**” chips

Semi–Custom Design

Using Pre–fabricated Chips

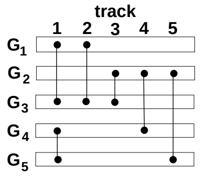
- fabricated up to the inter-connection phase
- reduces overall cost of manufacturing chips
- example: Programmable Logic Arrays (PLA)
  - gates are laid out in rows ($G_1, G_2, \dots, G_n$) with specified connections between certain pairs, G_i and G_j, given as ($< i, j >$)
  - connections are laid out in parallel tracks (columns)
  - no connections may overlap, not even at an endpoint
  - we want to **minimize** the number of tracks required

A Programmable Logic Array Example

Suppose we want the following Gate connections

$< 1,\ 3 >$ $< 1,\ 3 >$ $< 2,\ 3 >$
$< 2,\ 4 >$ $< 2,\ 5 >$ $< 4,\ 5 >$

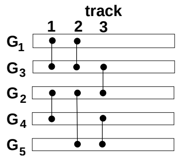
The layout below "**realizes**" these connections using 5 tracks



Minimizing the Layout

If we have the time and money, we can re-arrange rows and improve the routing phase. Notice that gates in rows 1 & 3 want to be together, as do gates in rows 2 & 3, and in 4 & 5

We get very good improvement: from 5 tracks down to just 3.



How is this modeled in a program? With a graph. We can use a force–directed algorithm, which acts like springs attached to the rows so those with many connections are more attracted than those with fewer.

Interval Graph

- **Interval Graph**: a graph G with a one–to–one correspondence between its vertices and a collection of intervals on the line such that two vertices of G are adjacent when the corresponding intervals overlap.
- Example applications: competition graph (used in ecology; species compete for survival), VLSI routing problems (PLA folding).

- Given the VLSI design problem of connecting the rows of gates:

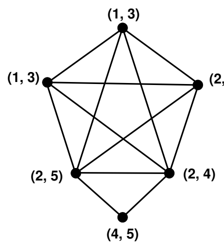
```
(1, 3)    (1, 3)    (2, 3)
(2, 4)    (2, 5)    (4, 5)
```

we can model the problem of determining the minimum number of tracks by finding the **Chromatic Number** of a related interval graph.

- Let vertices be the connection pairs, and consider them as intervals, for example: (1, 3) $\rightarrow$ [1..3]
- Let edges join intervals without overlap.
- The minimum number of tracks will be the **Chromatic Number** of the graph since intervals can share a track only if they do not overlap.

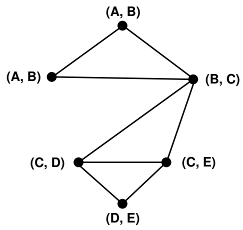
Consider: K_5 Subgraph

- A complete graph, K_5, requires 5 colors (and we cannot color it in fewer colors).
- Thus, we need at least 5 tracks.



Another Example

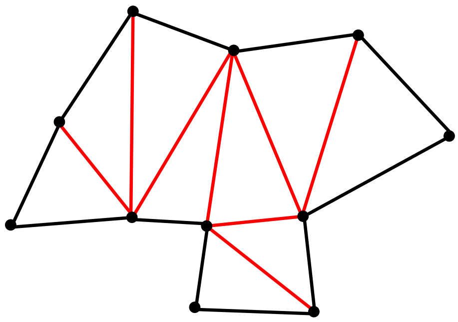
- The largest complete graph in the figure below is a triangle, therefore it requires 3 colors (and we cannot color it in 2 colors)
- Thus we need only 3 tracks when the rows are rearranged.



Sec. 2.4—Coloring Theorems

- **Polygon**: a planar graph consisting of a single circuit with edges drawn as straight lines.
- **Triangulation of a Polygon**: the process of adding a set of straight–line chords between pairs of vertices of the polygon so that all interior regions are bounded by a triangle.
- **Note**: Chords cannot cross each other nor the sides of the polygon.

Example of a Triangulated Polygon



Triangulation Theorem

Theorem 1. The vertices in a triangulation of a polygon can be 3–colored.

Proof is by induction on n, the number of edges in the polygon.

BC. Let n = 3.

Then the polygon is a triangle, and clearly can be 3–colored.

IH. (Strong induction) Assume any triangulated polygon with $4 \leq k < n$ boundary edges can be 3–colored for some arbitrary $n \geq 4$.

IS. Show a triangulated polygon, T, with n boundary edges can be 3–colored.

- Pick some chord edge $e = < v_i, v_j >$, which must exist since T has been triangulated.
- Since **all** chord edges connect vertices of the polygon, the chord edge e **splits** T into two smaller triangulated polygons, each of which can be 3–colored by the **IH**.
- In each coloring, v_i will have some color, and v_j will have some other color.
- Then the two subgraphs can be combined to yield a 3–coloring of the original polygon since, if need be, the coloring of one of the smaller polygons can be modified. **Note**: this 3–coloring is unique.

Application of the Theorem

- **Art Gallery Problem**: What are the **fewest** number of guards needed to watch paintings along the n walls of an art gallery?
- Guards must have direct line–of–sight to every point on the walls.
- A guard at a corner is assumed to be able to see the two walls that end at that corner, and the wall directly opposite the corner, if there is one.

Fisk's Corollary

Corollary: the Art Gallery Problem with n walls requires **at most** $\lfloor \frac{n}{3} \rfloor$ guards (where $\lfloor \ \rfloor$ is the floor function.)

Proof: Let the n walls form a polygon P with triangulation T. 3–color T and note each triangle will have a corner of each color. Pick one color, c, and place a guard at each corner colored c (1 in each triangle). Hence the sides (and thus all walls) of every triangle will be watched.

A polygon with n walls has n corners. If there are n corners and 3 colors, some color is used on $\lfloor \frac{n}{3} \rfloor$ or fewer corners.

Other Coloring Theorems

Notes:

- If a graph is bipartite, it is 2–colorable (and vice–versa)
- A graph is 2–colorable IFF all circuits have even length (this doesn't require the graph to be connected)
- Let $\chi(G)$ denote the chromatic number of G.

Theorem 2. If the graph G is not an odd circuit or a complete graph, then $\chi(G) \leq d$ where d is the maximum degree of a vertex in G. (This gives a usually poor upper bound on $\chi(G)$)

Theorem 3. For any positive integer k, there exists a triangle–free graph G with $\chi(G) = k$

Rather than color vertices, we can color edges so that all edges incident to the same vertex must have different colors.

Theorem 4 (Vizing's Theorem). If the maximum degree of a vertex in a graph G is d, then the **edge chromatic number** of G is either d or $d+1$.

Theorem 5. Every planar graph can be 5–colored

Note: in Tucker, Section 1.4, exercise 16, the reader was asked to prove: **Any connected planar graph has a vertex of degree at most 5**.

Theorem 5 Proof — by induction on the number of vertices.

BC. Let $1 \leq n \leq 5$.
Trivially, any such n vertex graph can be 5–colored.

IH. Assume for some arbitrary $n \geq 1$, that connected planar graphs with $n-1$ vertices can be 5–colored.

IS. Show a graph with n vertices can be 5–colored.

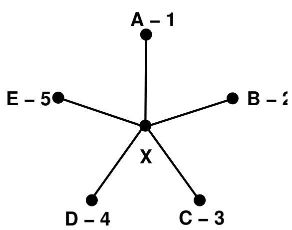
By the note, G must have a vertex, x, of degree at most 5. Delete x from G to obtain a graph, G', with $n-1$ vertices. By the `IH`, G' is 5–colorable.

Now, reconnect x to the graph and try to properly color x.

If x has degree ≤ 4, then simply assign a color to x which is different from any of its neighbors. The same coloring works if the degree of x is 5 and 2 or more of its neighbors has the same color.

There remains the case of how to color x if all 5 neighbors have different colors.

- Let us label the adjacent vertices A, B, C, D, and E, imposing a clock–wise ordering around X in a planar depiction of G.
- Let the colors 1—5 be assigned to vertices A—E in order.
- Consider vertex A, colored 1, and vertex C, colored 3.



Case 1.

If there is **no path** between them (other than through A), A may be recolored with color 3, all vertices adjacent to A which are 3 can be assigned 1, and so on.

This re-coloring will not affect C since there is no path from A to C, and furthermore, will only affect vertices reachable from A which are colored 1 or 3.

After the re-coloring, X may be colored 1.

Case 2.

There **exists a path** from A to C. This path either encompasses B, or it encompasses D, but not both, since G is planar.

Thus there can be no path between B and D (other than through A), so the same type of re-coloring may be applied to B (color 2), using D's color (4).

Thus allowing X to be colored with 2.

Hence, every planar graph can be 5–colored.

Tucker, Chapter 2 Overview

- Section 2.1 Euler cycles — cycles that traverse every edge exactly once. Determine existence with Euler's Theorem.
- Section 2.2 Hamilton circuits — circuits that visit every vertex exactly once. Determine existence by a laborious systematic search to try all possible ways of constructing a HC.
- Section 2.3 Graph coloring & some applications.
- Section 2.4 Graph coloring theory.