Mat3770 — Relations

Spring 2014

Student Responsibilities

- **Reading**: Textbook, Section 8.1, 8.3, 8.5
- **Assignments**:

  Sec 8.1 1a-d, 3a-d, 5ab, 16, 28, 31, 48bd
  Sec 8.3 1ab, 3ab, 5, 14a-c, 18(ab), 23, 26, 36
  Sec 8.5 2ad, 5, 15, 22, 35, 43a-c, 61
- **Attendance**: Spritefully Encouraged

Overview

- Sec 8.1 Relations and Their Properties
- Sec 8.3 Representing Relations
- Sec 8.5 Equivalence Relations

Section 8.1 — Relations and Their Properties

Binary Relations

- **Definition**: A **binary relation R** from a set A to a set B is a subset $R \subseteq A \times B$.
- Note: there are no constraints on relations as there are on functions.
- We have a common graphical representation of relations, a directed graph.

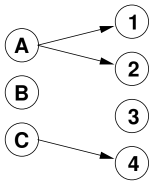
Directed Graphs

- **Definition**: A **Directed Graph (Digraph) D** from A to B is:
  1. a collection of **vertices V** $\subseteq A \cup B$, and
  2. a collection of **edges E** $\subseteq A \times B$
- If there is an ordered pair $e = < x, y >$ in R, then there is an **arc** or **edge** from x to y in D. (Note: $E = R$)
- The elements x and y are called the **initial** and **terminal** vertices of the edge e.

Relation Example

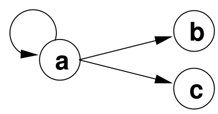
- Let A = { a, b, c },
- B = { 1, 2, 3, 4 }, and
- R be defined by the ordered pairs or edges:

  $$\{ < a, 1 >, < a, 2 >, < c, 4 > \}$$
- Then we can represent R by the digraph D:



Relation on a Single Set A

- **Definition**: A binary relation **R on a set A** is a subset of $A \times A$ or a relation from A to A.
- Let A = { a, b, c }
- R = $\{ < a, a >, < a, b >, < a, c > \}$
- Then a digraph representation of R is:



Notes

- An arc of the form $<x, x>$ on a digraph is called a **loop**.
- Question: How many binary relations are there on a set A?
  Another way to think of it:

  How many subsets are there of A $\times$ A?

Special Properties of Binary Relations

Given

1. A universe U
2. A binary relation R on a subset A of U

- **Definition**: R is **reflexive** IFF

$$\forall\, x \;\; [x \in A \;\rightarrow\; <x, x> \in R]$$

- Notes:
  - If $A = \emptyset$, then the implication is vacuously true
  - The void relation on an empty set is reflexive
  - If A is not void, then **all** vertices in the reflexive relation must have loops

Symmetric and Antisymmetric Properties

- **Definition**: R is **symmetric** IFF

$$\forall x \forall y[<x, y> \in R \;\rightarrow\; <y, x> \in R]$$

  Note: if there is an arc $<x, y>$, there must be an arc $<y, x>$

- **Definition**: R is **antisymmetric** IFF

$$\forall x \forall y[(<x, y> \in R) \wedge (<y, x> \in R) \;\rightarrow\; x = y]$$

  Note: If there is an arc from x to y, there cannot be one from y to x if x$\neq$ y.

  To prove a relation is antisymmetric, show logically that if $<x, y>$ is in R and x $\neq$ y, then $<y, x>$ is not in R.

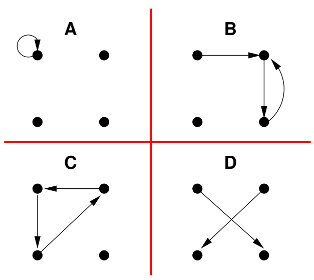
The Transitive Property

- **Definition**: R is **transitive** IFF

$$\forall x \forall y \forall z[(<x, y> \in R) \wedge (<y, z> \in R) \;\rightarrow\; <x, z> \in R]$$

  Note: If there is an arc from x to y and one from y to z, then there must be one from x to z.

  This is the most difficult property to check. We will develop algorithms to check this later.



R	reflexive	symmetric	antisymmetric	transitive
A		$\surd$	$\surd$	$\surd$
B				
C			$\surd$	
D			$\surd$	$\surd$

Combining Relations — Set Operations

- A very large set of potential questions! For example, let R1 and R2 be binary relations on a set A. Then we have questions of the form:

  If R1 has Property_1 and
  R2 has Property_2,
  does R1 $\star$ R2 have Property_3?

- For example, If R1 is symmetric and R2 is antisymmetric, does it follow that R1 $\cup$ R2 is transitive?

  If so, we need to prove it;
  otherwise, we can find a counterexample.

Another Example

- Let R1 and R2 be transitive on A. Does it follow that R1 $\cup$ R2 is transitive?
- Consider:
  - A = $\{ 1, 2 \}$
  - R1 = $\{ <1,2> \}$
  - R2 = $\{ <2,1> \}$
- Then R1 $\cup$ R2 = $\{ <1,2>, <2,1> \}$. which is **not** transitive. (Why not?)

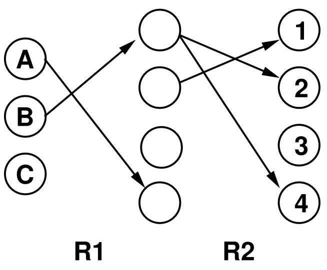
Composition of Relations

- **Definition**: Suppose
  - R1 is a relation from A to B
  - R2 is a relation from B to C

  Then the **composition of R2 with R1**, denoted R2 $\circ$ R1, is the relation from A to C:

  If $<x, y>$ is a member of R1 and
  $<y, z>$ is a member of R2 , then
  $<x, z>$ is a member of R2 $\circ$ R1
- For $<x, y>$ to be in the composite relation R2 $\circ$ R1, there must exist a y in B
- We read compositions right to left as in functions, applying R1 first, then R2 in this example.

Example of a Composite Relation

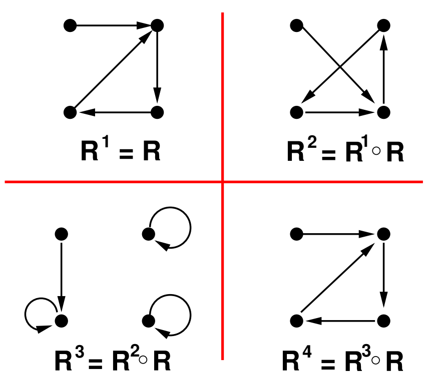


R2 $\circ$ R1 = $\{ <B,2>, <B,4> \}$

A Relation Composed with Itself

- **Definition**: Let R be a **binary relation** on A. Then the powers $R^n, n = 1, 2, 3, \ldots$ are defined recursively by:
  - **Basis**: $R^1 = R$
  - **Induction**: $R^{n+1} = R^n \circ R$
- Note: An ordered pair $<x, y>$ is in R^n IFF there is a **path** of length n from x to y following the arcs (in the direction of the arrows) in R.

Composites on R



A Very Important Theorem

R is transitive IFF $R^n \subseteq R$ for n > 0.

Proof ($\Rightarrow$): R transitive $\rightarrow R^n \subseteq$ R

Use a direct proof with proof by induction

- Assume R is transitive & show $R^n \subseteq$ R by induction

  Basis: Obviously true for n = 1

  Induction:
  - IH: Assume $R^k \subseteq R$ for some arbitrary k > 0
  - IS: Show $R^{k+1} \subseteq R$

    $R^{k+1} = R^k \circ R$, so if $<x, y>$ is in R^{k+1}, then there is a z such that $<x, z>$ is in R^k and $<z, y>$ is in R.

    But, since $R^k \subseteq$ R, $<x, z>$ is in R

    R is transitive, so $<x, y>$ is in R

    Since $<x, y>$ was an arbitrary edge, the result follows

Proof ($\Leftarrow$)

To complete the proof, we need to show:

$$R^n \subseteq R \rightarrow R \text{ is transitive}$$

Use the fact that $R^2 \subseteq R$ and the definition of transitivity. Proof left as an exercise...

Thus, (given a finished proof of the above) we have shown:

R is transitive IFF $R^n \subseteq R$ for n > 0

Section 8.3 — Representing Relations

Connection Matrices

- Let R be a relation from $A = \{ a_1, a_2, \ldots, a_m \}$ to $B = \{ b_1, b_2, \ldots, b_n \}$
- **Definition**: An $m \times n$ **connection matrix**, M, for R is defined by:

$$m_{i,j} = \begin{cases} 1 & \textit{if} \quad < a_i, b_j > \quad \in \quad R \\ 0 & \textit{otherwise} \end{cases}$$

Example

- Assume the rows are labeled with the elements of A and the columns are labeled with the elements of B.

  Let A = { a, b, c }, B = { e, f, g, h }, and
  $R = \{ < a, e >, < c, g > \}$

- Then the connection matrix M for R is: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
- **Note**: The order of the elements of A and B is important!

Theorem. let R be a binary relation on a set A and let M be its connection matrix. Then:

- R is **reflexive** IFF $M_{i,i} = 1$ for all $1 \leq i \leq |A|$
- R is **symmetric** IFF M is a symmetric Matrix: $M = M^T$
- R is **antisymmetric** if $M_{ij} = 0$ or $M_{ji} = 0$ for all $i \neq j$

Combining Connection Matrices — Join

- **Definition**: The **join** of two matrices, M_1 and M_2, denoted $M_1 \vee M_2$, is the component–wise Boolean "or" of the two matrices.
- **Fact**: If M_1 is the connection matrix for R_1, and M_2 is the connection matrix for R_2, then the join of M_1 and M_2, $M_1 \vee M_2$, is the connection matrix for $R_1 \cup R_2$

Combining Connection Matrices — Meet

- **Definition**:The **meet** of two matrices, M_1 and M_2, denoted $M_1 \wedge M_2$, is the component–wise Boolean "and" of the two matrices.
- **Fact**: If M_1 is the connection matrix for R_1, and M_2 is the connection matrix for R_2, then the meet of M_1 and M_2, $M_1 \wedge M_2$, is the connection matrix for $R_1 \cap R_2$

Finding Connection Matrix Combinations

Given the connection matrix for two relations, how does one find the connection matrix for:

- The **complement**: $(A \times A) - R$
- The **relative complement**: $R_1 - R_2$ and $R_2 - R_1$
- The **symmetric difference**: $R_1 \oplus R_2 = (R_1 - R_2) \cup (R_2 - R_1)$

The Composition

- **Definition**: Let M_1 be the connection matrix for R_1, and M_2 be the connection matrix for R_2.

  Then the **Boolean product** of these two matrices, denoted $M_1 \otimes M_2$, is the connection matrix for the composition of R_2 with R_1, $R_2 \circ R_1$

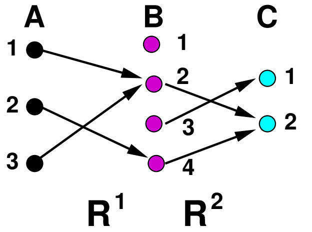
$$(M_1 \otimes M_2)_{ij} = \vee_{k=1}^{n}[(M_1)_{ik} \wedge (M_2)_{kj})]$$

- Why? In order for there to be an arc $< x, z >$ in the composition, then there must be an arc $< x, y >$ in R_1 and an arc $< y, z >$ in R_2 for some y.

Notes on Composition

- The Boolean product checks all possible y's. If at least one such path exists, that is sufficient.
- The matrices M_1 and M_2 must be **conformable**: the number of columns of M_1 must equal the number of rows of M_2.
- If M_1 is $m \times n$ and M_2 is $n \times p$, then $M_1 \otimes M_2$ is $m \times p$

Composition Example — $R_2 \circ R_1$



$$M_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad M_1 \otimes M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} (M_1 \otimes M_2)_{12} &= [(M_1)_{11} \wedge (M_2)_{12}] \vee \\ &\quad [(M_1)_{12} \wedge (M_2)_{22}] \vee \\ &\quad [(M_1)_{13} \wedge (M_2)_{32}] \vee \\ &\quad [(M_1)_{14} \wedge (M_2)_{42}] \vee \\ &= [0 \wedge 0] \vee [1 \wedge 1] \vee [0 \wedge 0] \vee [0 \wedge 1] \\ &= 1 \end{aligned}$$

- There is an arc in R_1 from node 1 in A to node 2 in B
- There is an arc in R_2 from node 2 in B to node 2 in C
- Hence, there is an arc in $R_2 \circ R_1$ from node 1 in A to node 2 in C.
- **A useful result**: $M_{R^n} = (M_R)^n$

Digraphs

Given the digraphs for R_1 and R_2, describe how to find the digraphs for:

- **Union**: $R_2 \cup R_1$
- **Intersection**: $R_2 \cap R_1$
- **Relative Complement**: $R_2 - R_1$
- **Boolean Product**: $R_2 \otimes R_1$
- **Complement**: $A \times A - R_1$
- **Symmetric Difference**: $R_1 \oplus R_2$

Section 8.5 — Equivalence Relations

Wherein we define new types of important relations by grouping properties of relations together.

Definition: A relation R on a set A is an **equivalence relation** IFF R is:

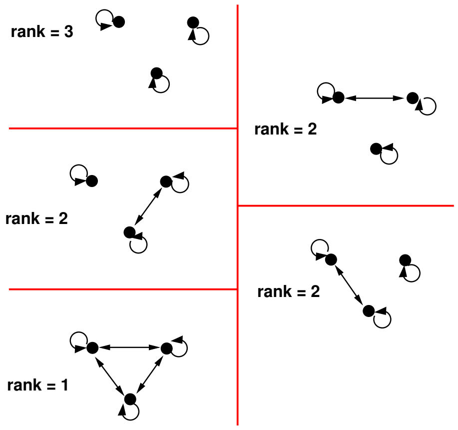
- **reflexive**
- **symmetric**, and
- **transitive**

Equivalence Relations in Digraphs

Equivalence relations are easily recognized in digraphs

- The set of all elements related to a particular element forms a subgraph in which **all** self–loops and arcs are present between the included vertices.
- I.e., the digraph (or subdigraph) representing the subset will be a **complete** digraph (or subdigraph).
- The number of such subsets is called the **rank** of the equivalence relation.

All Equivalence Relations on a Set with 3 Elements



Equivalence Classes

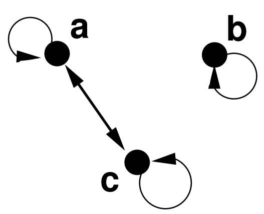
- Each complete subset is called an **equivalence class**.
- A bracket around an element means the equivalence class in which the element lies.

$$[x] = \{y| < x, y > \in R\}$$

- The element in the bracket is called a **representative** of the equivalence class — we could have chosen any element in that class.
- Three ways to say "in the **same equivalence class**":

aRb	$[a] = [b]$	$[a] \cap [b] \neq \emptyset$

Example — Equivalence Classes



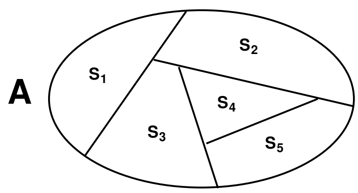
$[a] = \{a, c\}$, $[c] = \{a, c\}$, $[b] = \{b\}$

rank $= 2$

Partitions

- **Definition**: Let $S_1, S_2, \ldots, S_n$ be a collection of subsets of A. Then the collection forms a **partition of A** if the subsets are **non–empty**, **disjoint**, and **exhaust A**.

$S_i \neq \emptyset$	$S_i \cap S_j = \emptyset$ if $i \neq j$	$\bigcup S_i = A$

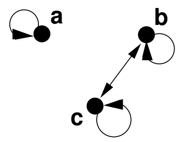


Theorem. The equivalence classes of an equivalence relation R partition the set A into disjoint nonempty subsets whose union is the entire set.

The partition is denoted A/R and is called:

- the **quotient set** or
- the **partition of A induced by R**, or
- **A modulo R**

Examples

1. The set of integers such that aRb IFF a = b or a = −b
2. The natural numbers mod any integer:
   For example, N mod 3 divides the natural numbers into 3 equivalence classes: $[0]_3$, $[1]_3$, $[2]_3$
3. 

$$[a] = \{a\}, \quad [b] = \{b, c\}, \quad [c] = \{b, c\}$$

rank = 2

Theorem. Let R be an equivalence relation on A. Then either

$$[a] = [b]$$

or

$$[a] \cap [b] = \emptyset$$

Why?

Review — $R \subseteq A \times A$

- **reflexive**: $(a, a) \in R \ \forall \ a \in A$
- **symmetric**: $(b, a) \in R \leftrightarrow (a, b) \in R$ *for* $a, b \in A$
- **antisymmetric**:
  $(b, a) \in R$ *and* $(a, b) \in R$, *then* $a = b$ *for* $a, b \in A$
- **transitive**:
  $(a, b) \in R$ *and* $(b, c) \in R$, *then* $(a, c) \in R$ *for* $a, b, c \in A$

Transitive Closure

- A **path** in a digraph is a sequence of connected edges.
- A path has **length** n, where n is the number of edges in the path.
- A **circuit** or **cycle** is a path that begins and ends at the same vertex.
- The **connectivity relation R*** is the set of all pairs <a, b> such that there is a path between a and b in the relation R.

  This is also called the **transitive closure** of R.

Theorem. If R_1 and R_2 are equivalence relations on A, then $R_1 \cap R_2$ is an equivalence relation on A.

Proof outline:

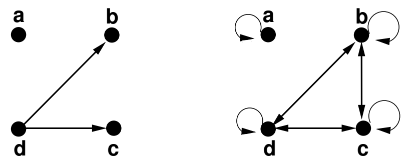
It would suffice to show that the **intersection** of:

- reflexive relations is reflexive
- symmetric relations is symmetric, and
- transitive relations is transitive

Reflexive, Symmetric, Transitive Closure

Definition. Let R be a relation on A. Then the **reflexive**, **symmetric**, **transitive** closure of R, denoted **tsr(R)**, is an **equivalence relation** induced by R on A.

Example:



R **tsr(R), rank = 2**

A = [a]∪[b] = {a}∪ {b, c, d}

A/R = { {a}, {b, c, d} }

Theorem. tsr(R) is an equivalence relation.

Proof. We must be careful and show that tsr(R) is still symmetric and reflexive.

- Since we only add arcs (rather than delete arcs) when computing closures, it must be that tsr(R) is reflexive since all loops $< x, x >$ on the digraph must be present when constructing r(R).

- If there is an arc $< x, y >$, then the symmetric closure of r(R) ensures there is an arc $< y, x >$.

- We may now argue that if we construct the transitive closure of sr(R) and we add an edge $< x, z >$ because there is a path from x to z, then there must also exist a path from z to x, and hence we also must add an edge $< z, x >$. Hence the transitive closure of sr(R) is symmetric.