Week 5 — Student Responsibilities

- Reading: Edge Counting, Planarity
  (See Syllabus schedule)

- Hwk from Tucker – 2.4

- Hwk from Rosen – 9.8

- Attendance Sprightly Encouraged
What is the Chromatic Number of this Graph?
An Example Coloring
Graph Coloring Applications—VLSI Chip Design

**VLSI**: Very Large Scale Integrated Chip Design—the “brains” of a computer

**Gates**: logical sub-circuits in a computer chip which are composed of electronic switches

Possibilities in chip manufacturing:

- Most expensive: *Custom fabricated* chips
- Medium expense: *Semi–custom* chips
- Least expensive: “*Off–the–Shelf*” chips
Semi–Custom Design

Using Pre–fabricated Chips

- fabricated up to the inter-connection phase
- reduces overall cost of manufacturing chips
- example: Programmable Logic Arrays (PLA)
  - gates are laid out in rows \((G_1, G_2, \ldots, G_n)\) with specified connections between certain pairs, \(G_i\) and \(G_j\), given as \((<i, j>)\)
  - connections are laid out in parallel tracks (columns)
  - no connections may overlap, not even at an endpoint
  - we want to minimize the number of tracks required
A Programmable Logic Array Example

Suppose we want the following Gate connections

\(< 1, 3 > \quad < 1, 3 > \quad < 2, 3 > \\
< 2, 4 > \quad < 2, 5 > \quad < 4, 5 >\)

The layout below “realizes” these connections using 5 tracks
Minimizing the Layout

If we have the time and money, we can re-arrange rows and improve the routing phase. Notice that gates in rows 1 & 3 want to be together, as do gates in rows 2 & 3, and in 4 & 5.

We get very good improvement: from 5 tracks down to just 3.

How is this modeled in a program? With a graph. We can use a force–directed algorithm, which acts like springs attached to the rows so those with many connections are more attracted than those with fewer.
Interval Graph

- **Interval Graph**: a graph $G$ with a one-to-one correspondence between its vertices and a collection of intervals on the line such that two vertices of $G$ are adjacent when the corresponding intervals overlap.

- Example applications: competition graph (used in ecology; species compete for survival), VLSI routing problems (PLA folding).
Given the VLSI design problem of connecting the rows of gates:

(1, 3)  (1, 3)  (2, 3)
(2, 4)  (2, 5)  (4, 5)

we can model the problem of determining the minimum number of tracks by finding the **Chromatic Number** of a related interval graph.
Let vertices be the connection pairs, and consider them as intervals, for example: \((1, 3) \rightarrow [1..3]\)

Let edges join intervals without overlap.

The minimum number of tracks will be the **Chromatic Number** of the graph since intervals can share a track only if they do not overlap.
Consider: $K_5$ Subgraph

- A complete graph, $K_5$, requires 5 colors (and we cannot color it in fewer colors).

- Thus, we need at least 5 tracks.
Another Example

- The largest complete graph in the figure below is a triangle, therefore it requires 3 colors (and we cannot color it in 2 colors)

- Thus we need only 3 tracks when the rows are rearranged.
Sec. 2.4—Coloring Theorems

- **Polygon**: a planar graph consisting of a single circuit with edges drawn as straight lines.

- **Triangulation of a Polygon**: the process of adding a set of straight-line chords between pairs of vertices of the polygon so that all interior regions are bounded by a triangle.

- **Note**: Chords cannot cross each other nor the sides of the polygon.
Example of a Triangulated Polygon
**Theorem 1.** The vertices in a triangulation of a polygon can be 3–colored.

Proof is by induction on $n$, the number of edges in the polygon.

**BC.** Let $n = 3$.

Then the polygon is a triangle, and clearly can be 3–colored.

**IH.** (Strong induction) Assume any triangulated polygon with $4 \leq k < n$ boundary edges can be 3–colored for some arbitrary $n \geq 4$. 
IS. Show a triangulated polygon, $T$, with $n$ boundary edges can be 3–colored.

- Pick some chord edge $e = < v_i, v_j >$, which must exist since $T$ has been triangulated.

- Since all chord edges connect vertices of the polygon, the chord edge $e$ splits $T$ into two smaller triangulated polygons, each of which can be 3–colored by the IH.

- In each coloring, $v_i$ will have some color, and $v_j$ will have some other color.

- Then the two subgraphs can be combined to yield a 3–coloring of the original polygon since, if need be, the coloring of one of the smaller polygons can be modified. **Note:** this 3–coloring is unique.
Application of the Theorem

- **Art Gallery Problem**: What are the fewest number of guards needed to watch paintings along the $n$ walls of an art gallery?

- Guards must have direct line–of–sight to every point on the walls.

- A guard at a corner is assumed to be able to see the two walls that end at that corner, and the wall directly opposite the corner, if there is one.
Fisk’s Corollary

**Corollary**: the Art Gallery Problem with $n$ walls requires at most $\lfloor \frac{n}{3} \rfloor$ guards (where $\lfloor \rfloor$ is the floor function.)

**Proof**: Let the $n$ walls form a polygon $P$ with triangulation $T$. 3–color $T$ and note each triangle will have a corner of each color. Pick one color, $c$, and place a guard at each corner colored $c$ (1 in each triangle). Hence the sides (and thus all walls) of every triangle will be watched.

A polygon with $n$ walls has $n$ corners. If there are $n$ corners and 3 colors, some color is used on $\lfloor \frac{n}{3} \rfloor$ or fewer corners.
Other Coloring Theorems

**Notes:**

- If a graph is bipartite, it is 2–colorable (and vice–versa)

- A graph is 2–colorable IFF all circuits have even length (this doesn’t require the graph to be connected)

- Let $\chi(G)$ denote the chromatic number of $G$.

**Theorem 2.** If the graph $G$ is not an odd circuit or a complete graph, then $\chi(G) \leq d$ where $d$ is the maximum degree of a vertex in $G$. (This gives a usually poor upper bound on $\chi(G)$)
**Theorem 3.** For any positive integer $k$, there exists a triangle–free graph $G$ with $\chi(G) = k$

Rather than color vertices, we can color edges so that all edges incident to the same vertex must have different colors.

**Theorem 4** (Vizing’s Theorem). If the maximum degree of a vertex in a graph $G$ is $d$, then the edge chromatic number of $G$ is either $d$ or $d + 1$. 
Theorem 5. Every planar graph can be 5–colored

**Note:** in Tucker, Section 1.4, exercise 16, the reader was asked to prove: *Any connected planar graph has a vertex of degree at most 5.*

Theorem 5 Proof — by induction on the number of vertices.

**BC.** Let $1 \leq n \leq 5$.
Trivially, any such $n$ vertex graph can be 5–colored.

**IH.** Assume for some arbitrary $n \geq 1$, that connected planar graphs with $n - 1$ vertices can be 5–colored.

**IS.** Show a graph with $n$ vertices can be 5–colored.
By the note, $G$ must have a vertex, $x$, of degree at most 5. Delete $x$ from $G$ to obtain a graph, $G'$, with $n - 1$ vertices. By the IH, $G'$ is 5–colorable.

Now, reconnect $x$ to the graph and try to properly color $x$.

If $x$ has degree $\leq 4$, then simply assign a color to $x$ which is different from any of its neighbors. The same coloring works if the degree of $x$ is 5 and 2 or more of its neighbors has the same color.

There remains the case of how to color $x$ if all 5 neighbors have different colors.
Let us label the adjacent vertices $A$, $B$, $C$, $D$, and $E$, imposing a clock-wise ordering around $X$ in a planar depiction of $G$.

Let the colors 1—5 be assigned to vertices $A$—$E$ in order.

Consider vertex $A$, colored 1, and vertex $C$, colored 3.

```
A − 1
E − 5
B − 2
D − 4
C − 3
X
```
Case 1.

If there is **no path** between them (other than through $A$), $A$ may be recolored with color 3, all vertices adjacent to $A$ which are 3 can be assigned 1, and so on.

This re-coloring will not affect $C$ since there is no path from $A$ to $C$, and furthermore, will only affect vertices reachable from $A$ which are colored 1 or 3.

After the re-coloring, $X$ may be colored 1.
Case 2.

There **exists a path** from $A$ to $C$. This path either encompasses $B$, or it encompasses $D$, but not both, since $G$ is planar.

Thus there can be no path between $B$ and $D$ (other than through $A$), so the same type of re-coloring may be applied to $B$ (color 2), using $D$'s color (4).

Thus allowing $X$ to be colored with 2.

**Hence, every planar graph can be 5–colored.**
Tucker, Chapter 2 Overview

- **Section 2.1** Euler cycles — cycles that traverse every edge exactly once. Determine existence with Euler’s Theorem.

- **Section 2.2** Hamilton circuits — circuits that visit every vertex exactly once. Determine existence by a laborious systematic search to try all possible ways of constructing a HC.

- **Section 2.3** Graph coloring & some applications.

- **Section 2.4** Graph coloring theory.