

Network Flow

Conservation

Max Flow

flow

Cancellation

Cut

Ford-
Fulkerson

Residual
Network

Augmenting
Path

Max-flow
Min-cut

Matching

Mat 3770 Network Flows

Spring 2014

4.2 Network Flows

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Network Flow

Conservation

Max Flow

flow

Cancellation

Cut

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Residual
Network

Augmenting
Path

Max-flow
Min-cut

Matching

- A directed graph can model a **flow network** where some material (e.g., widgets, current, ...) is **produced** or enters the network at a **source** and is **consumed** at a **sink**.
- Production and consumption are at a **steady rate**, which is the same for both.
- The **flow** of the material at any point in the system is the rate at which the material moves through it.

Modeling

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Conservation

Max Flow

flow

Cancellation

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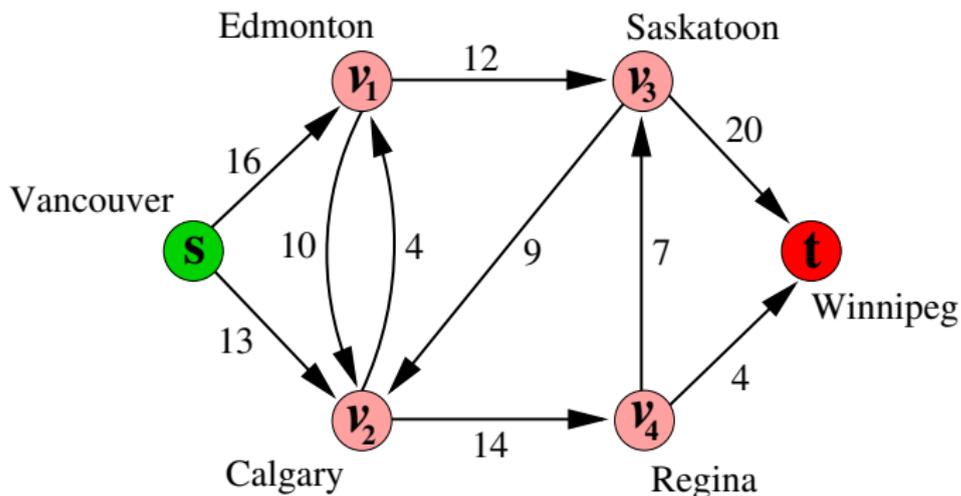
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Max-flow
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- Flow networks can be used to model:
 - liquids through pipe
 - parts through an assembly line
 - current through electrical networks
 - info through communication networks
- Each directed edge is a **conduit** for the material.
- Each conduit has a stated **capacity** given as a maximum rate at which the material can flow through the conduit. (e.g., 200 barrels of oil per hour.)

A Network Flow Example



A flow network for the Lucky Duck Puck factory, located in Vancouver, with warehouse in Winnipeg. Each edge is labeled with its capacity.

from: *Introduction to Algorithms*, by Cormen, Leiserson, & Rivest

Example Details

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Conservation

Max Flow

flow

Cancellation

Cut

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Network

Augmenting
Path

Max-flow
Min-cut

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- The Lucky Duck Company has a **factory** (source s) in Vancouver that manufactures hockey pucks.
- They have a **warehouse** (sink t) in Winnipeg that stores them.
- They lease space on trucks from another firm to ship the pucks from the factory to the warehouse — with **capacity** $c(u, v)$ crates per day between each pair of cities u and v .

Goal: determine p , the largest number of crates per day that can be shipped, and then produce this amount — there's no sense in producing more pucks than they can ship to their warehouse.

- The rate at which pucks are shipped along any truck route is a **flow**.

- **Maximum flow** determines p , the maximum number of crates per day that can be shipped.

- The pucks leave the factory at the rate of p crates per day, and p crates must arrive at the warehouse each day:



- Shipping time is not a concern, only the flow of p crates/day.
- **Capacity constraints** are given by the restriction that the flow $f(u, v)$ from city u to city v be at most $c(u, v)$ crates per day.
- In a **steady state**, the number of crates entering and the number leaving an intermediate city must be equal.

Flow Conservation

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Network Flow

Conservation

Max Flow

flow

Cancellation

Cut

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Fulkerson

Residual
Network

Augmenting
Path

Max-flow
Min-cut

Matching

- Vertices are conduit **junctions**. Other than the **source** and **sink**, material flows through the vertices **without collecting** in them.
- Hence, the rate at which material **enters** a vertex must **equal** the rate at which it **leaves** the vertex.
- This property is called **flow conservation**, and is similar in concept to Kirchhoff's Current Law concerning electrical current.

Maximum-flow

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Network Flow

Conservation

Max Flow

flow

Cancellation

Cut

Ford-
Fulkerson

Residual
Network

Augmenting
Path

Max-flow
Min-cut

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The **Maximum-flow** problem is the simplest problem concerning flow networks:

What is the greatest rate at which material can be shipped from source to sink without violating any capacity constraints?

Assumptions

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Conservation

Max Flow

flow

Cancellation

Cut

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Residual
Network

Augmenting
Path

Max-flow
Min-cut

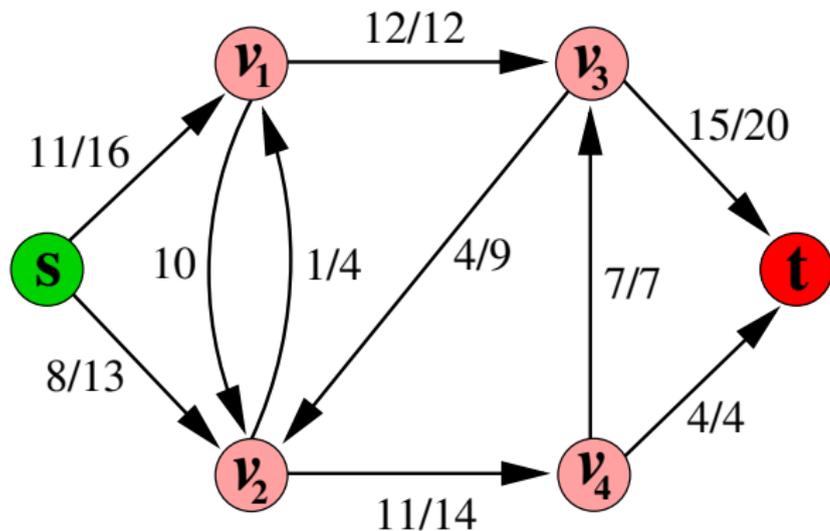
Matching

A **flow network**, $G = (V, E)$, is a directed graph in which each edge $(u, v) \in E$ has a non-negative **capacity** $c(u, v) \geq 0$.

If $(u, v) \notin E$, we assume $c(u, v) = 0$.

$\forall x \in E$, $\text{In}(x)$ and $\text{Out}(x)$ are the edges into and out of vertex x .

The integer $c(e)$ associated with edge e is a capacity or upper bound.



A flow f with value $|f| = 19$.

Only positive net flows (crates shipped) are shown.

If $f(u, v) > 0$, edge (u, v) is labeled by $f(u, v)/c(u, v)$.
If $f(u, v) \leq 0$, edge (u, v) is labeled only by its capacity.

Tucker: a–z (Source to Sink) Flow ϕ

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- An **a–z flow** ϕ in a directed network N is an integer–valued function ϕ defined on each edge e .

- $\phi(e)$ is the flow in e , together with a **source** vertex a and a **sink** vertex z satisfying the following three conditions:

1. **Capacity Constraint:** $0 \leq \phi(e) \leq c(e)$

We don't want *backflow*, nor to exceed any edge's capacity

2. $\phi(e) = 0$ if $e \in \text{IN}(a)$ or $e \in \text{OUT}(z)$

We want the flow to go from source to sink, not vice-versa

3. **Flow Conservation:**

For $x \neq a$ or z , $\sum_{e \in \text{IN}(x)} \phi(e) = \sum_{e \in \text{OUT}(x)} \phi(e)$

For every vertex other than the source and sink, the flow into and out of that vertex must be equal

Network Flow

Conservation

Max Flow

flow

Cancellation

Cut

Ford-
Fulkerson

Residual
Network

Augmenting
Path

Max-flow
Min-cut

Matching

Flow Networks

A **flow** in G is a real-valued function $f : V \times V \rightarrow \mathbb{R}$ that satisfies the following three properties:

1. **Capacity constraint**: the flow along an edge cannot exceed its capacity:

$$\forall u, v \in V, f(u, v) \leq c(u, v)$$

2. **Skew symmetry**: the flow from a vertex u to a vertex v is the negative of the flow in the reverse direction:

$$\forall u, v \in V, f(u, v) = -f(v, u)$$

3. **Flow conservation**: the net flow of a vertex (other than the source or sink) is 0:

$$\forall u \in V - \{s, t\}, \sum_{v \in V} f(u, v) = 0$$

- The quantity $f(u, v)$, which can be positive, negative, or zero, is called the **flow** from vertex u to vertex v .

- The **value** of a flow f is defined as:

$$|f| = \sum_{v \in V} f(s, v) = \sum_{v \in V} f(v, t).$$

I.e., the total flow **out of the source** or **into the sink**.

- In the **maximum-flow problem**, we are given a flow network G with source s and sink t , and we wish to find a flow of maximum value from s to t .

1. (By skew symmetry) The flow from a vertex to itself is 0, since for all $u \in V$, we have $f(u, u) = -f(u, u)$
2. (By skew symmetry) We can rewrite the flow-conservation property as the total flow into a vertex is 0:

$$\forall v \in V - \{s, t\}, \quad \sum_{u \in V} f(u, v) = 0$$

Flow In Equals Flow Out

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1. The total positive flow **entering** a vertex v is defined by

$$\sum_{u \in V, f(u, v) > 0} f(u, v)$$

2. The total positive flow **leaving** a vertex v is defined by

$$\sum_{u \in V, f(v, u) > 0} f(v, u)$$

3. The positive net flow entering a vertex (other than the source or sink) must equal the positive net flow leaving the vertex.

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Conservation

Max Flow

flow

Cancellation

Cut

Ford-
Fulkerson

Residual
Network

Augmenting
Path

Max-flow
Min-cut

Matching

There can be no flow between u and v
if there is no edge between them.

If there is no edge between u and v , i.e., $(u, v) \notin E$ and $(v, u) \notin E$, then:

- If there is no edge, then the capacity is zero.
And by the Capacity Constraint, the flows must be ≤ 0 .

$$c(u, v) = c(v, u) = 0 \Rightarrow f(u, v) \leq 0, f(v, u) \leq 0$$

- By skew symmetry, the flow must be zero.

$$f(u, v) = -f(v, u) \Rightarrow f(u, v) = f(v, u) = 0$$

Cancellation

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Network
Flows

Network Flow

Conservation

Max Flow

flow

Cancellation

Cut

Ford-
Fulkerson

Residual
Network

Augmenting
Path

Max-flow
Min-cut

Matching

- **Cancellation** allows us to represent the shipments between two cities by a positive flow along at most one of the two edges between the corresponding vertices.
- If there is zero or negative flow from one vertex to another, no shipments need be made in that direction.
- Any situation in which pucks are shipped in both directions between two cities can be transformed using **cancellation** into an **equivalent** situation in which pucks are shipped **only** in the direction of **positive flow**.
- No constraints are violated since the net flow between the two vertices is the same.

Cancellation in General

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Network Flow

Conservation

Max Flow

flow

Cancellation

Cut

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Residual
Network

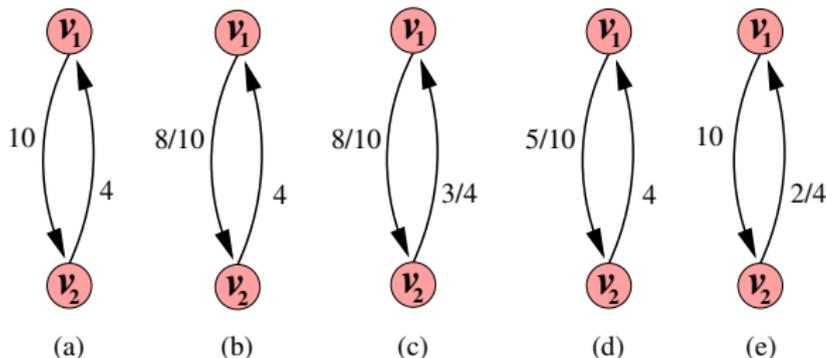
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- Suppose edge (u, v) has flow value $f(u, v)$
- Now, increase the flow on the on edge (v, u) by some amount d
- Mathematically, this operation must decrease $f(u, v)$ by d
- Conceptually, we can think of these d units as canceling d units of flow that are already on edge (u, v)

Cancellation Example



- (a) Vertices v_1 and v_2 , with $c(v_1, v_2) = 10$ and $c(v_2, v_1) = 4$
- (b) Net flow when 8 crates per day are shipped from v_1 to v_2 .
- (c) An additional shipment of 3 crates per day from v_2 to v_1 .
- (d) By canceling flow going in opposite directions, we can represent the situation in (c) with positive flow in one direction only.
- (e) Another 7 crates per day shipped from v_2 to v_1 results in a net of 2 crates per day from v_2 to v_1 .

Flow Cuts

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Network Flow

Conservation

Max Flow

flow

Cancellation

Cut

Ford-
Fulkerson

Residual
Network

Augmenting
Path

Max-flow
Min-cut

Matching

- A **cut**(S, T) of flow network $G = (V, E)$, is a partition of V into S and $T = V - S$, such that **source** $\in S$ and **sink** $\in T$.
- If f is a flow, then the **net flow** across the $cut(S, T)$ is defined to be $f(S, T)$.
- The **capacity** of the $cut(S, T)$ is $c(S, T)$.

Cut Example

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Network Flow

Conservation

Max Flow

flow

Cancellation

Cut

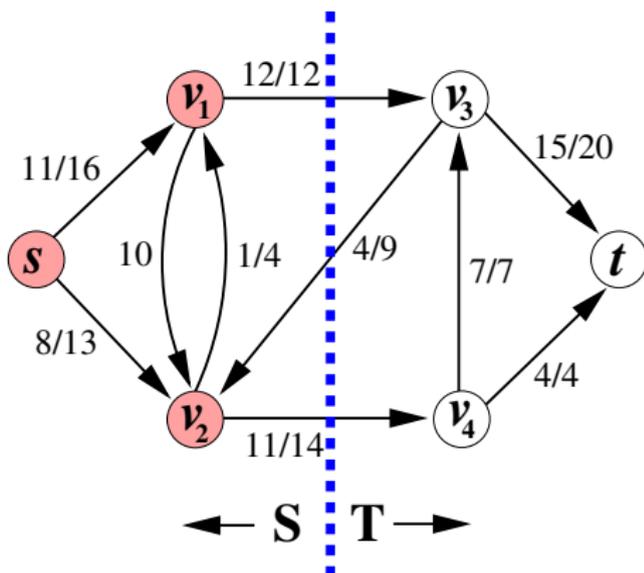
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Network

Augmenting
Path

Max-flow
Min-cut

Matching



cut($\{s, v_1, v_2\}, \{v_3, v_4, t\}$) has **net flow**:

$$f(v_1, v_3) + f(v_2, v_3) + F(v_2, v_4) = 12 - 4 + 11 = 19.$$

And, its **capacity** is: $c(v_1, v_3) + c(v_2, v_4) = 12 + 14 = 26$.

Ford–Fulkerson Method

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Each iteration will increase the flow until
the maximum is reached

Ford–Fulkerson Method (G, s, t)

initialize flow f to 0

while there exists an augmenting path p

augment flow f along p

return f

Network Flow

Conservation

Max Flow

flow

Cancellation

Cut

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Residual
Network

Augmenting
Path

Max-flow
Min-cut

Matching

Ford–Fulkerson Solution

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Flows

Network Flow

Conservation

Max Flow

flow

Cancellation

Cut

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Network

Augmenting
Path

Max-flow
Min-cut

Matching

The Ford–Fulkerson Method for solving the maximum–flow problem depends on three key concepts:

1. residual networks
2. augmenting paths
3. cuts

Residual Networks

Given a flow network and a flow, the **residual network** consists of edges that can accommodate more net flow.

- Suppose we have a flow network $G = (V, E)$, with source s and sink t .
- Let f be a flow in G ; consider a pair of vertices $u, v \in V$.
- The amount of **additional** flow we can push from u to v before exceeding the capacity $c(u, v)$ is the **residual capacity** of (u, v) given by:

$$c_f(u, v) = c(u, v) - f(u, v)$$

- For example, if $c(u, v) = 16$ and $f(u, v) = 11$, we can ship $c_f(u, v) = 5$ more units before we exceed capacity.

- When the **flow** is **negative**, the **residual capacity** is **greater than the capacity**

E.g., $c(u, v) = 16$, $f(u, v) = -4$, so $c_f(u, v) = 20$

- This can be interpreted to mean:
 1. There is a flow of 4 units from $v \rightarrow u$, which we can cancel by pushing a flow of 4 units from $u \rightarrow v$.
 2. We can then push another 16 units from $u \rightarrow v$ before violating the capacity constraint on edge (u, v) .
 3. We have thus pushed an additional 20 units of flow, starting with a flow $f(u, v) = -4$, before reaching the capacity constraint.

- Given a flow network $G = (V, E)$ and a flow f , the **residual network** of G induced by f is $G_f = (V, E_f)$, where:

$$E_f = \{(u, v) \in V \times V : c_f(u, v) > 0\}$$

I.e., the Residual network consists of edges that can accommodate more net flow.

- Each edge of the residual network (**residual edge**), can admit a strictly positive new flow.
- Notice (u, v) may be a residual edge in E_f even if it was not an edge in E — i.e., it could be the case that $E_f \not\subseteq E$.

- Because an edge (u, v) can appear in a residual network only if at least one of (u, v) and (v, u) appears in the original network, we have the bound: $|E_f| \leq 2|E|$.
- Observation: the residual network G_f is itself a flow network with capacities given by c_f .
- **Lemma.** Let $G = (V, E)$ be a flow network with source s and sink t , and let f be a flow in G . Let G_f be the residual network of G induced by f , and let f' be a flow in G_f . Then the flow sum $f + f'$ is a flow in G with value:

$$|f + f'| = |f| + |f'|$$

This lemma shows how a flow in a residual network relates to a flow in the original flow network.

Augmenting Paths

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Network Flow

Conservation

Max Flow

flow

Cancellation

Cut

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Fulkerson

Residual
Network

**Augmenting
Path**

Max-flow
Min-cut

Matching

- Given a flow network $G = (V, E)$ and a flow f , an **augmenting path** p is a simple path from s to t in the residual network G_f .
- By the definition of the residual network, each edge (u, v) on an augmenting path admits some additional positive flow from u to v without violating the capacity constraint on the edge.
- The maximum amount of flow we can ship along the edges of an augmenting path p is the **residual capacity** of p given by:

$$c_f(p) = \min\{c_f(u, v) : (u, v) \text{ is on } p\}$$

Lemma. Let $G = (V, E)$ be a flow network, let f be a flow in G , and let p be an augmenting path in G_f . Define a function $f_p : V \times V \rightarrow \mathfrak{R}$ by:

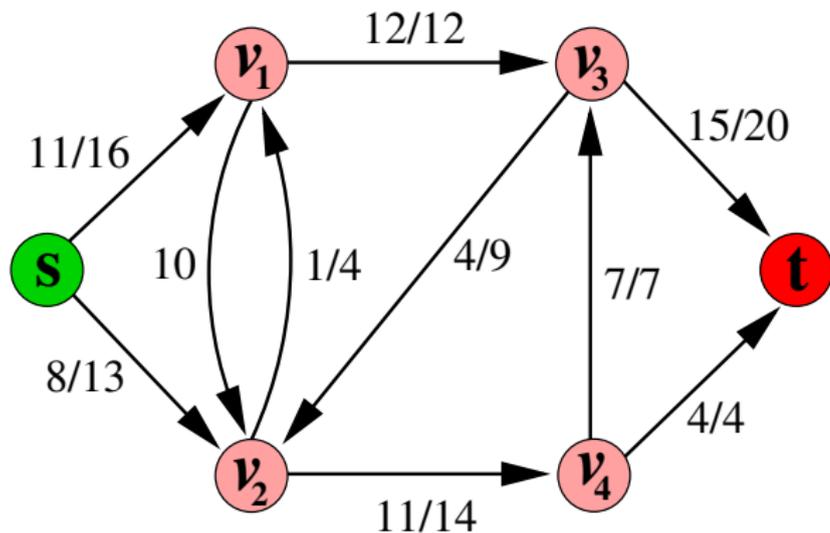
$$f_p(u, v) = \begin{cases} c_f(p) & : \text{if } (u, v) \text{ is on } p \\ -c_f(p) & : \text{if } (v, u) \text{ is on } p \\ 0 & : \text{otherwise} \end{cases}$$

- If we add f_p to f , we get another flow in G whose value is closer to the maximum.
- **Corollary.** Let $G = (V, E)$ be a flow network, let f be a flow in G , and let p be an augmenting path in G_f . Let f_p be defined as in the previous lemma. Define a function $f' : V \times V \rightarrow \mathfrak{R}$ by $f' = f + f_p$. Then f' is a flow in G with value:

$$|f'| = |f| + |f_p| > |f|$$

Flow Network G and Flow F

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Conservation

Max Flow

flow

Cancellation

Cut

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Fulkerson

Residual
Network

Augmenting
Path

Max-flow
Min-cut

Matching

Residual network G_f

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Network Flow

Conservation

Max Flow

flow

Cancellation

Cut

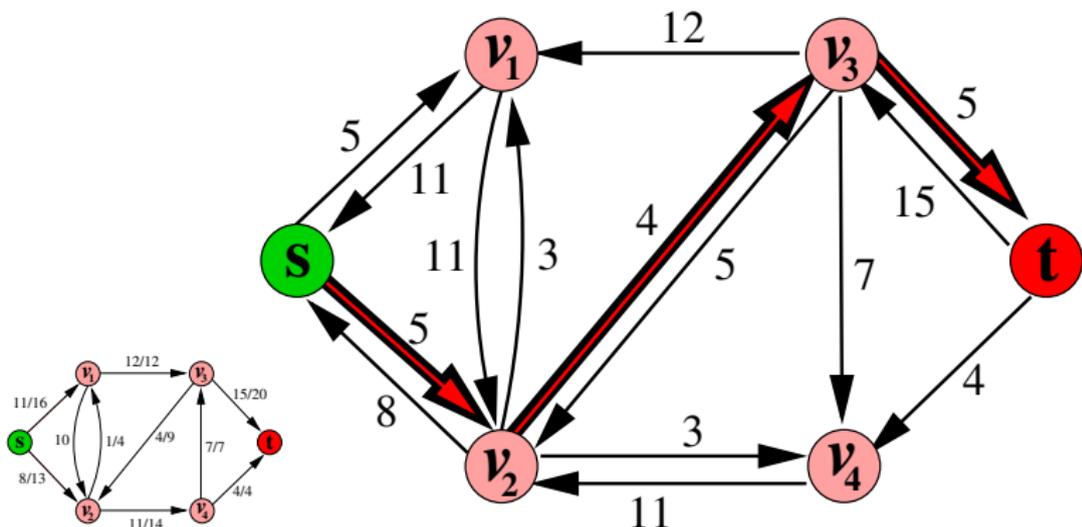
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Residual
Network

Augmenting
Path

Max-flow
Min-cut

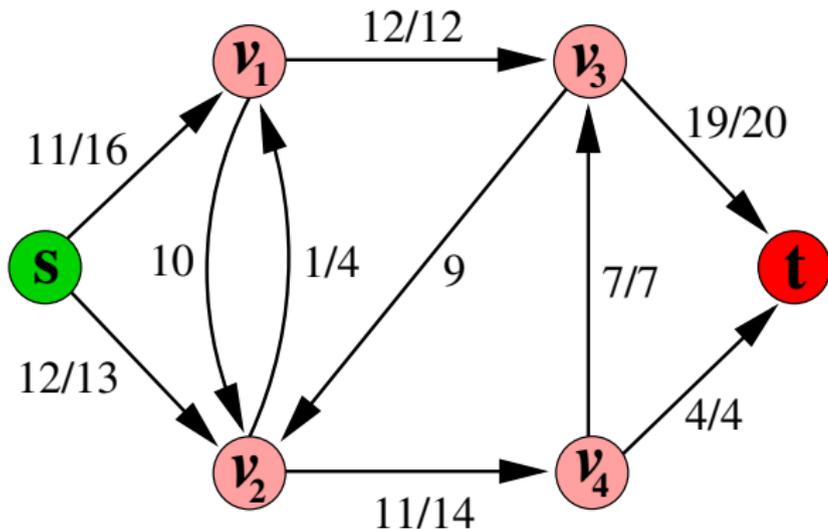
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Residual network G_f with augmenting path p indicated;
its residual capacity is $c_f(p) = c(v_2, v_3) = 4$

Resulting Flow After Augmenting

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Resulting flow in G from augmenting path p
by its residual capacity

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Conservation

Max Flow

flow

Cancellation

Cut

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Network

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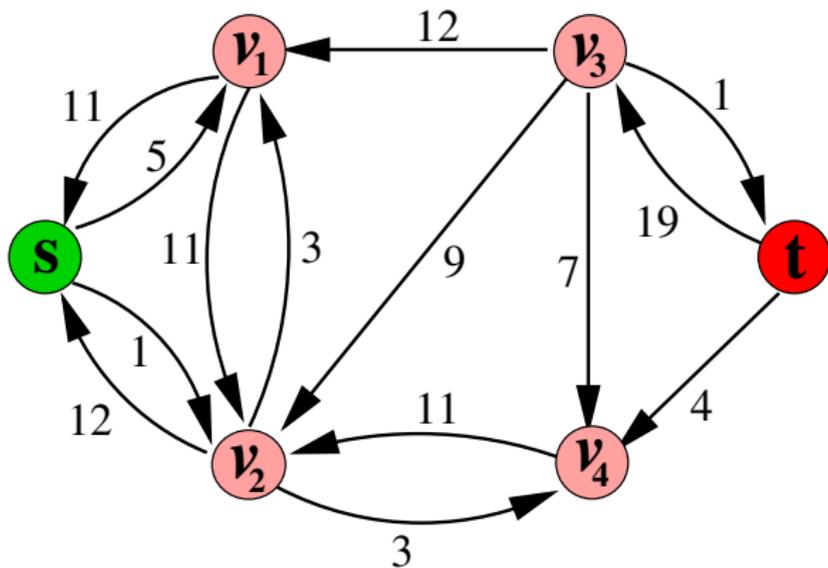
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Min-cut

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New Residual Network

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Residual network induced by the flow in (c)

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Conservation

Max Flow

flow

Cancellation

Cut

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Fulkerson

Residual
Network

Augmenting
Path

Max-flow
Min-cut

Matching

Cuts of Flow Networks

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Conservation

Max Flow

flow

Cancellation

Cut

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Fulkerson

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Path**

Max-flow
Min-cut

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The Ford–Fulkerson method repeatedly augments the flow along augmenting paths until a maximum flow has been found.

The max–flow min–cut theorem tells us a flow is maximum
IFF
its residual network contains no augmenting path.

The Max-flow Min-cut Theorem

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Conservation

Max Flow

flow

Cancellation

Cut

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Fulkerson

Residual
Network

Augmenting
Path

Max-flow
Min-cut

Matching

- **Lemma.** Let f be a flow in a flow network G with source s and sink t , and let (S, T) be a cut of G . Then the net flow across (S, T) is $f(S, T) = |f|$.
- **Corollary.** The value of any flow f in a flow network G is bounded from above by the capacity of any cut of G .
- **Max-flow min-cut Theorem.** If f is a flow in a flow network $G = (V, E)$ with with source s and sink t , then the following conditions are **equivalent**:
 1. f is a maximum flow in G
 2. The residual network G_f contains no augmenting paths
 3. $|f| = c(S, T)$ for some cut (S, T) of G

The Basic Ford–Fulkerson Algorithm

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Flows

Network Flow

Conservation

Max Flow

flow

Cancellation

Cut

Ford-
Fulkerson

Residual
Network

Augmenting
Path

Max-flow
Min-cut

Matching

- Each iteration of the Ford–Fulkerson method, finds *any* augmenting path p and augments flow f along p by the residual capacity $c_f(p)$
- The Ford–Fulkerson method computes the maximum flow in a graph $G = (V, E)$ by updating the net flow $f(u, v)$ between each pair of vertices that are connected by an edge. If u and v are not connected by an edge, assume $f(u, v) = 0$
- Assume the capacity from u to v is provided by a constant–time function $c(u, v)$ with:

$$c(u, v) = 0 \text{ if } (u, v) \notin E$$

- The residual capacity is computed as $c_f(u, v) = c(u, v) - f(u, v)$, represented by $c(p)$ in the algorithm.

Ford-Fulkerson (G, s, t)

1 for each edge (u, v) in $E[G]$

2 $f[u, v] = 0$

3 $f[v, u] = 0$

4 while there exists a path (s, t) in the
 residual network G_f

5 $cf(p) = \min\{cf(u, v) : (u, v) \text{ in } p\}$

6 for each edge (u, v) in p

7 $f[u, v] = f[u, v] + cf(p)$

8 $f[v, u] = -f[u, v]$

Lines 1 – 3 initialize the flow f to 0

The while loop repeatedly finds an augmenting path p in G_f ,
augments flow f along p by the residual capacity $cf(p)$

When no augmenting path exists, flow f is a maximum flow.

Time Analysis — Assuming Integer Capacities

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- Initialization, lines 1 – 3: $\Theta(|E|)$
- The while loop of lines 4 – 8 is executed at most $O(|f^*|)$ times, where f^* is the maximum flow found by the algorithm, since the flow value increases by at least one unit in each iteration.
- Let $m(p)$ = time to find a path with minimum residual capacity, and max = the maximum number of edges in such a path. Then the overall run time is:

$$O(E + |f^*|(m(p) + max))$$

Network Flow

Conservation

Max Flow

flow

Cancellation

Cut

Ford-
Fulkerson

Residual
Network

Augmenting
Path

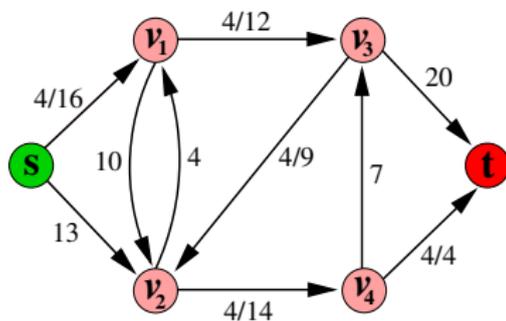
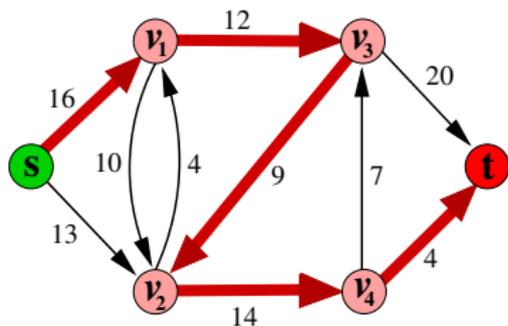
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Min-cut

Matching

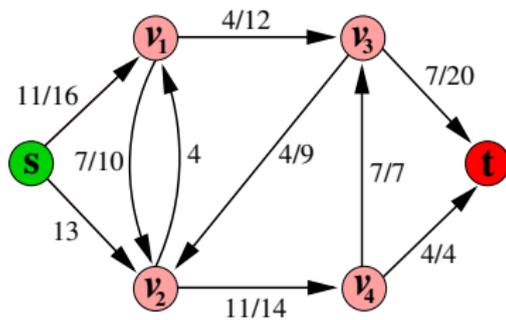
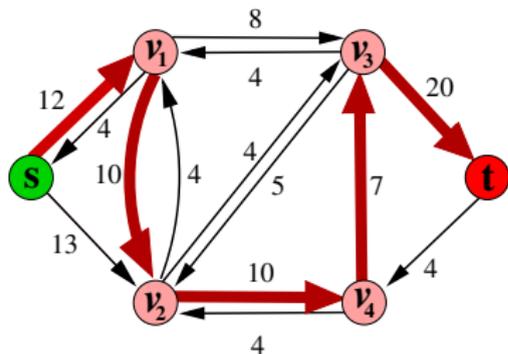
Successive iterations of the Ford–Fulkerson algorithm are shown on the next set of slides.

On each slide, the first graph (on left) shows the residual network G_f with an augmenting path p .

The second graph (right) shows the new flow f that results from adding f_p to f .



(a) The residual network (left) is the input network G
On right is the new, resulting flow



(b) The next augmenting path, with new flow

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Conservation

Max Flow

flow

Cancellation

Cut

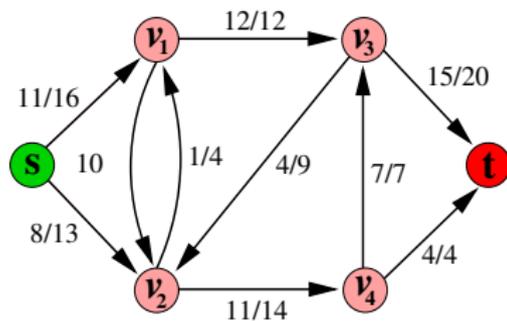
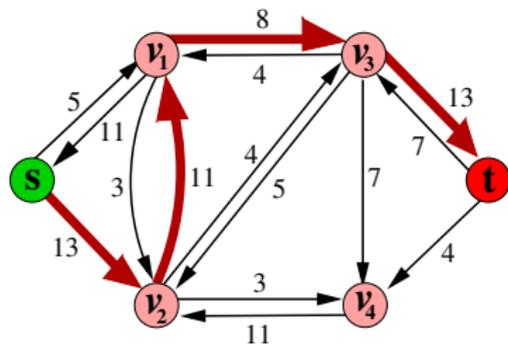
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Fulkerson

Residual
Network

Augmenting
Path

Max-flow
Min-cut

Matching



(c) The next augmenting path with, new flow

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Conservation

Max Flow

flow

Cancellation

Cut

Ford-
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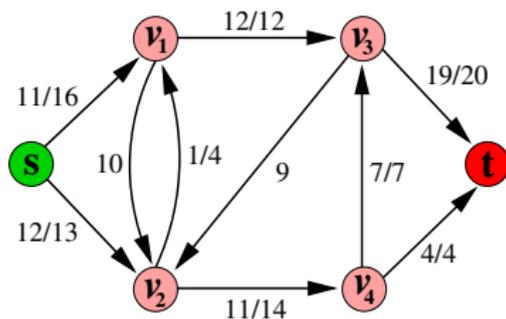
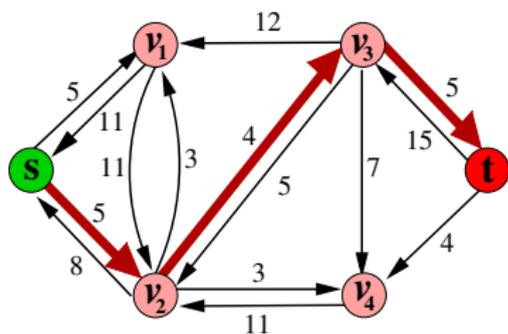
Residual
Network

Augmenting
Path

Max-flow
Min-cut

Matching

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Network
Flows



(d) The last augmenting path, with new flow

Network Flow

Conservation

Max Flow

flow

Cancellation

Cut

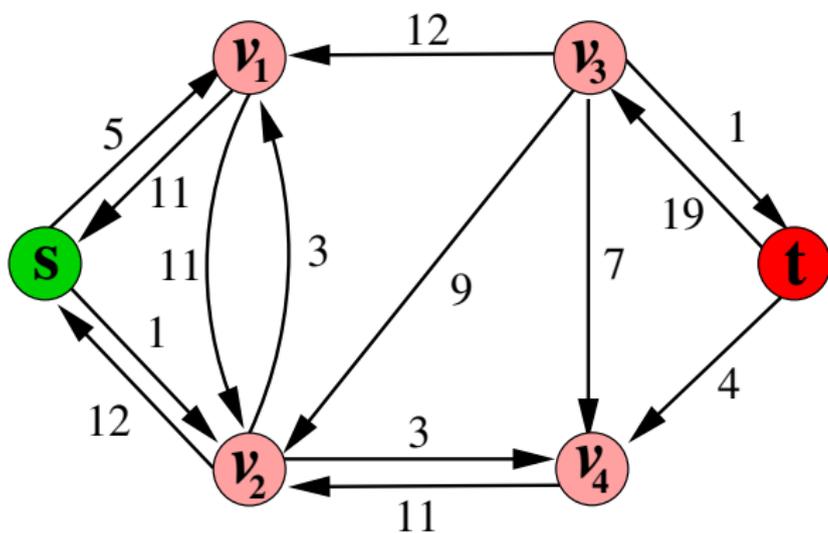
Ford-
Fulkerson

Residual
Network

Augmenting
Path

Max-flow
Min-cut

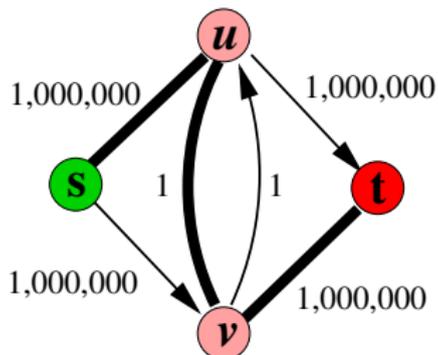
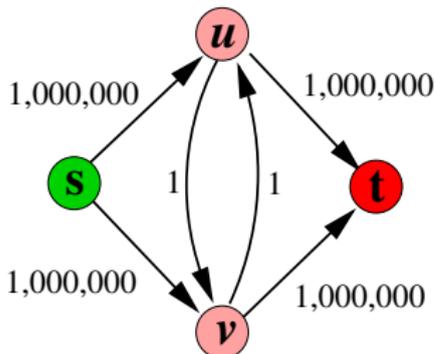
Matching



(e) No augmenting paths remain;
flow shown in (d) is maximum flow

Worst Case Example

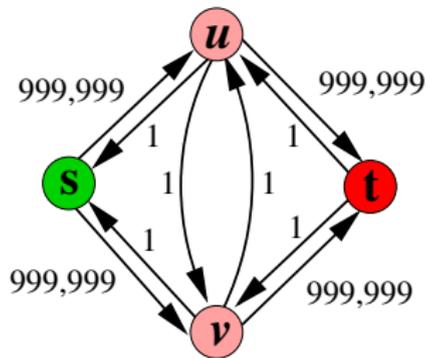
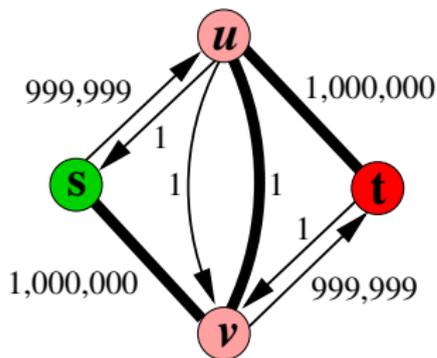
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(a) A flow network for which the algorithm can take $\Theta(E|f^*|)$ time

An augmenting path with residual capacity 1 is shown.

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Conservation
Max Flow
flow
Cancellation
Cut
Ford-
Fulkerson
Residual
Network
Augmenting
Path
Max-flow
Min-cut
Matching



(b) Another augmenting path with residual capacity 1, with the resulting residual network.

Eventually, the maximum flow of $|f^*| = 2,000,000$ will be reached.

Edmonds–Karp Algorithm

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Max Flow

flow

Cancellation

Cut

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Network

Augmenting
Path

Max-flow
Min-cut

Matching

- The bound on the Ford–Fulkerson algorithm can be improved if the computation of the augmenting path p (in line 4) is implemented with a breadth–first search.
- That is, if the augmenting path is a *shortest* path from s to t in the residual network, where each edge has unit distance (weight).
- The algorithm will then run in $O(VE^2)$ time.

4.3 Algorithmic Matching

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Max Flow

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Cancellation

Cut

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Fulkerson

Residual
Network

Augmenting
Path

Max-flow
Min-cut

Matching

- Some combinatorial problems can easily be cast as maximum-flow problems.
- One example: finding a maximum matching in a bipartite graph.
- The problem: Given an undirected graph $G = (V, E)$, a **matching** is a subset of edges $M \subseteq E \ni \forall v \in V$, at most one edge of M is incident on v .

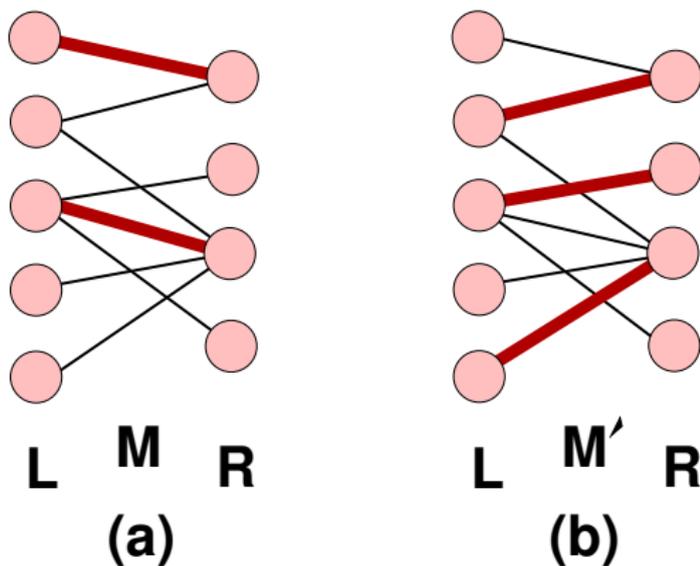


Fig 1. Bipartite graph $G = (V, E)$, vertex partition $V = L \cup R$
(a) A matching with cardinality 2
(b) A maximum matching with cardinality 3

- Vertex $v \in V$ is said to be **matched** by matching M if some edge in M is incident on v ; otherwise v is **unmatched**.
- A **maximum matching** is a matching of maximum cardinality: a matching $M \ni$ for any matching M' , we have $|M| \geq |M'|$.
- We are interested in finding maximum matchings in bipartite graphs.
- Assume the vertex set can be partitioned into $V = L \cup R$, where L and R are disjoint and all edges in E go between L and R .

Recast as a Flow Network Problem

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Conservation

Max Flow

flow

Cancellation

Cut

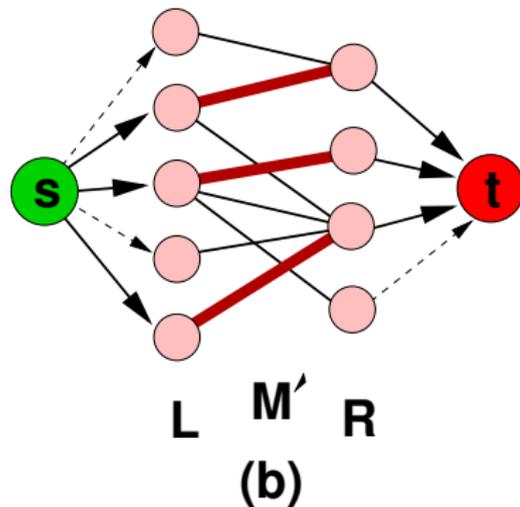
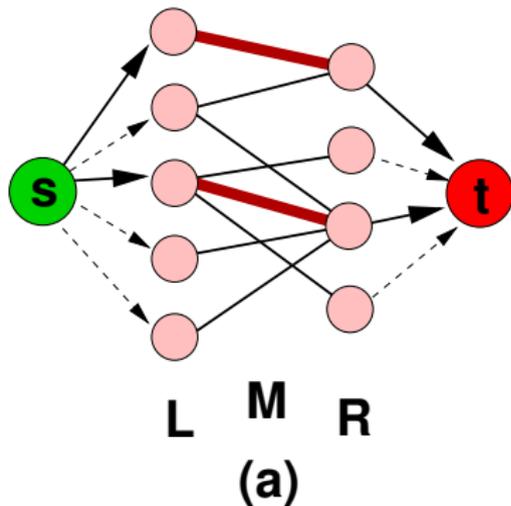
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Residual
Network

Augmenting
Path

Max-flow
Min-cut

Matching



A Practical Application

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Network Flow

Conservation

Max Flow

flow

Cancellation

Cut

Ford-
Fulkerson

Residual
Network

Augmenting
Path

Max-flow
Min-cut

Matching

- One (of many) practical application: matching a **set L of machines** with a **set R of tasks** to be performed simultaneously.
- The edge $\langle u, v \rangle \in E$ indicates a particular machine $u \in L$ is capable of performing a particular task $v \in R$
- A maximum matching provides work for as many machines as possible.

Finding a Maximum Bipartite Matching

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Conservation

Max Flow

flow

Cancellation

Cut

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Fulkerson

Residual
Network

Augmenting
Path

Max-flow
Min-cut

Matching

- We can use the Ford–Fulkerson method to find a maximum matching in an undirected bipartite graph, $G = (V, E)$, in time polynomial in $|V|$ and $|E|$.
- The trick is to construct a flow network in which flows correspond to matchings.

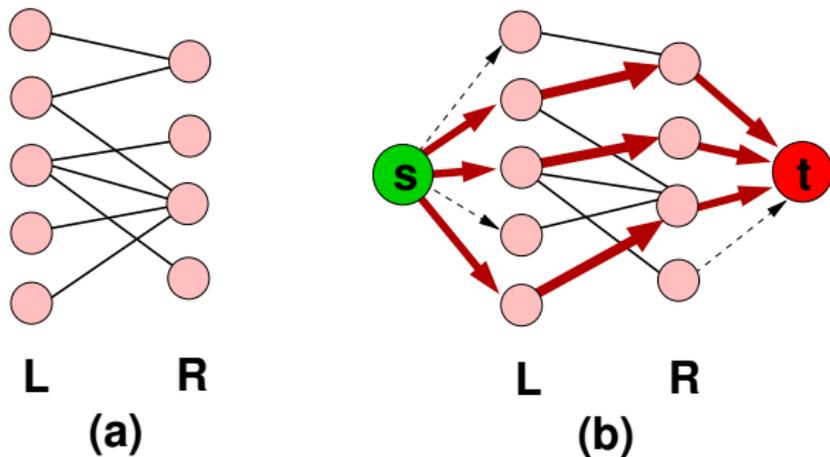


Fig. 2. The flow network corresponding to a bipartite graph.

(a) Bipartite graph with vertex partition from Fig. 1.

(b) Corresponding flow network with a maximum flow shown.

Each edge has unit capacity. Large arrows have a flow of 1, and all other edges carry no flow. The large arrows from L to R correspond to those in a max matching of the bipartite graph.

The **corresponding flow network** $G' = (V', E')$ for the bipartite graph G is defined as:

- Let the source s and sink t be new vertices not in V , and let $V' = V \cup \{s, t\}$.
- If the vertex partition of G is $V = L \cup R$, the directed edges of G' are given by:

$$E' = \{ \langle s, u \rangle : u \in L \} \cup \\ \{ \langle u, v \rangle : u \in L, v \in R, \text{ and} \\ \langle u, v \rangle \in E \} \cup \\ \{ \langle v, t \rangle : v \in R \}$$

- I.e., the edges from the source node to nodes in L , the original edges from L to R , and the edges from the nodes in R to the sink node.
- We then assign unit capacity to each of these edges in E'

- A matching in G corresponds directly to a flow in the corresponding flow network G' .
- A flow f on a flow network $G = (V, E)$ is **integer-valued** if $f(u, v)$ is an integer $\forall \langle u, v \rangle \in V \times V$.
- **Lemma.** Let $G = (V, E)$ be a bipartite graph with vertex partition $V = I \cup R$, and let $G' = (V', E')$ be its corresponding flow network.

If M is a matching in G , then there is an integer-valued flow f in G' with value $|f| = |M|$.

Conversely, if f is an integer-valued flow in G' , then there is a matching M in G with cardinality $|M| = |f|$.

- Intuitively, a maximum matching in a bipartite graph corresponds to a maximum flow in its corresponding flow network.
- We can compute a maximum matching in a bipartite graph by finding a maximum-flow in its flow network.
- The only possible problem: the maximum-flow algorithm might return a flow which consists of non-integral amounts (which would **not** lead to a good match).

- The following theorem shows that if we use the Ford–Fulkerson method, this difficulty cannot arise.
- **Theorem 1. (Integrality theorem).** If the capacity function c takes on only integral values, then the maximum flow f produced by the Ford–Fulkerson method has the property $|f|$ is integer-valued.

Moreover, for all vertices u and v , the value of $f(u, v)$ is an integer.

Proof is by induction on the number of iterations.

Corollary (to Lemma 1). The cardinality of a maximum matching in a bipartite graph G is the value of a maximum flow in its corresponding flow network G' .

Proof is by contradiction.