| **Reading**: Textbook, Section 7.1 & 7.2 |
| **Assignments**: Sec 7.1, 7.2 |
| **Attendance**: De–Lightfully Encouraged |

### Week 12 Overview

- Sec 7.1 Recurrence Relations
- Sec 7.2 Solving Linear Recurrence Relations
A recursive definition of a sequence specifies one or more initial terms plus a rule for determining subsequent terms from those that precede them.

A recurrence relation for the sequence \( \{a_n\} \) is an equation that expresses \( a_n \) in terms of one or more of the previous terms of the sequence, namely, \( a_0, a_1, \ldots, a_{n-1} \), for all integers \( n \) with \( n \geq n_0 \), where \( n_0 \) is a nonnegative integer.

The initial conditions for a sequence specify the terms that precede the first term where the recurrence relation takes effect.
A sequence is called a **solution** of a recurrence relation if its terms satisfy the recurrence relation.

A **Recurrence Relation** is a way to define a function by an expression involving the same function.
Modeling with Recurrence Relations – Rabbits

- A pair of rabbits (one of each sex) is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair produces another pair each month.

- **Fibonacci Numbers** (Pairs of Rabbits)
  
  \[ F(0) = 1, \quad F(1) = 1, \quad F(n) = F(n-1) + F(n-2) \]

- If we wish to compute the 120\textsuperscript{th} Fibonacci Number, \( F(120) \), we could compute \( F(0) \), \( F(1) \), \( F(2) \), \( \ldots \), \( F(118) \), and \( F(119) \) to arrive at \( F(120) \).

- Thus, to compute \( F(\mathbb{k}) \) in this manner would take \( \mathbb{k} \) steps.
It would be more convenient, not to mention more efficient, to have an explicit or closed form expression to compute $F(n)$.

Actually, for Fibonacci numbers, it’s:

$$F(n) = \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^n$$

\(\forall\) natural numbers \(n \geq 1\)
General problem: A person makes a deposit (principle) into a savings account which yields a yearly interest rate, compounded annually. How much will be in the account after 30 years?

Let \( P_n \) represent the amount in the account after \( n \) years.

Since \( P_n \) will equal the amount after \( n - 1 \) years plus interest, the sequence \( \{ P_n \} \) satisfies the recurrence relation:

\[
P_n = P_{n-1} + rP_{n-1} = (1 + r)P_{n-1}
\]
Recurrence Relations – Compound Interest

- We can use an iterative approach to find a formula for $P_n$:
  
  $P_1 = (1 + r)P_0$
  
  $P_2 = (1 + r)P_1 = (1 + r)^2P_0$
  
  $P_3 = (1 + r)P_2 = (1 + r)^3P_0$
  
  $\vdots$
  
  $P_n = (1 + r)P_{n-1} = (1 + r)^n P_0$

- Let’s assume $10,000 was deposited at 11% interest rate, compounded annually, for 30 years.

- Then $P_{30} = (1.11)^{30}10,000 = $228,922.97

- See other examples of modeling with RR in textbook.
Section 7.2 — Solving Recurrence Relations

- Recurrence relations which express the terms of a sequence as a **linear combination of previous terms** can be explicitly solved in a systematic way.

- **Definition** A **linear homogeneous recurrence relation of degree k with constant coefficients** is a recurrence relation of the form:

  \[ a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} \]

  where \( c_1, c_2, \ldots, c_k \) are real numbers, and \( c_k \neq 0 \)
- **Linear**: the right-hand side is a sum of multiples of the previous terms of the sequence.

- **Homogeneous**: no terms occur that are not multiples of the $a_j$’s

- **Constant Coefficients**: the coefficients of all the terms of the sequence are constants (rather than functions dependent on $n$)

- **Degree**: is $k$ because $a_n$ is expressed in terms of the previous $k$ terms of the sequence.
A sequence satisfying the recurrence relation in the definition is uniquely determined by this recurrence relation and the $k$ initial conditions:

$$a_0 = C_0, \quad a_1 = C_1, \quad \ldots, \quad a_{k-1} = C_{k-1},$$
Examples of linear homogeneous recurrence relations:

\[ P_n = 3P_{n-1} \quad \text{degree one} \]
\[ f_n = f_{n-1} + f_{n-2} \quad \text{degree two} \]
\[ a_n = a_{n-5} \quad \text{degree five} \]

Examples which are not linear homogeneous recurrence relations:

\[ a_n = a_{n-1} + a^2_{n-2} \quad \text{not linear} \]
\[ H_n = 2H_{n-1} + 2 \quad \text{not homogeneous} \]
\[ B_n = nB_{n-5} \quad \text{doesn’t have constant coefficient} \]
Solving Linear Homogeneous Recurrence Relations with Constant Coefficients

**Idea:** look for solutions of the form $a_n = r^n$, where $r$ is a constant.

**Note:** $a_n = r^n$ is a solution of the recurrence relation:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

if and only if

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \ldots + c_k r^{n-k}$$
Divide both sides of the previous equation by $r^{n-k}$, and subtract the right-hand side:

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_{k-1} r - c_k = 0$$

This is the **characteristic equation** of the recurrence relation.

**Note:** The sequence $\{a_n\}$ with $a_n = r^n$ is a solution IFF $r$ is a solution to the characteristic equation.
The solutions of the characteristic equation are called the **characteristic roots** of the recurrence relation.

They can be used to create an explicit formula for all the solutions of the recurrence relation.
Characteristic Roots

**Theorem 1.** Let $c_1$ and $c_2$ be real numbers. Suppose that

$$r^2 - c_1 r - c_2 = 0$$

has two distinct roots, $r_1$ and $r_2$. Then the sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

for $n = 0, 1, 2, \ldots$, where $\alpha_1$ and $\alpha_2$ are constants.
Solving Recurrence Relations, Example I

Let: \( a_0 = 2, \quad a_1 = 7, \) and \( a_n = a_{n-1} + 2a_{n-2} \)

We see that \( c_1 = 1 \) and \( c_2 = 2 \)

**Characteristic Equation:** \( r^2 - r - 2 = 0 \)

**Roots:** \( r = 2 \) and \( r = -1 \)

Thus, the sequence \( \{a_n\} \) is a solution to the recurrence relation IFF

\[
a_n = \alpha_1 2^n + \alpha_2 (-1)^n
\]

for some constants \( \alpha_1 \) and \( \alpha_2 \)
From the initial conditions, it follows that:

\[
\begin{align*}
a_0 &= 2 = \alpha_1 (2^0) + \alpha_2 (-1)^0 \\
a_1 &= 7 = \alpha_1 (2^1) + \alpha_2 (-1)^1
\end{align*}
\]

Solving these two equations yields:

\[
\alpha_1 = 3 \quad \text{and} \quad \alpha_2 = -1
\]

Therefore, the **solution** to the recurrence relation and initial conditions is the sequence \( \{a_n\} \) with:

\[
a_n = 3(2)^n - (-1)^n
\]
Solving Recurrence Relations, Example II

Let: \( F_0 = 0, \quad F_1 = 1, \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \)

We see that \( c_1 = 1 \) and \( c_2 = 1 \)

**Characteristic Equation:** \( r^2 - r - 1 = 0 \)

**Roots:** \( r = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad r = \frac{1 - \sqrt{5}}{2} \)

Thus, it follows that the Fibonacci numbers are given by

\[
F_n = \alpha_1 \left( \frac{1 + \sqrt{5}}{2} \right)^n + \alpha_2 \left( \frac{1 - \sqrt{5}}{2} \right)^n
\]

for some constants \( \alpha_1 \) and \( \alpha_2 \)
Solving Recurrence Relations, Example II — Cont.

From the initial conditions, it follows that:

\[ F_0 = 0 = \alpha_1 + \alpha_2 \]
\[ F_1 = 1 = \alpha_1 \left( \frac{1+\sqrt{5}}{2} \right) + \alpha_2 \left( \frac{1-\sqrt{5}}{2} \right) \]

Solving these two equations yields:

\[ \alpha_1 = \frac{1}{\sqrt{5}} \quad \text{and} \quad \alpha_2 = -\frac{1}{\sqrt{5}} \]

Therefore, the solution to the recurrence relation and initial conditions is the sequence \( \{F_n\} \) with:

\[ F_n = \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^n \]
Solving Recurrence Relations, Example III

Let: \( a_0 = 1, \ a_1 = 1, \) and \( a_n = 2a_{n-1} + 3a_{n-2} \)

We see that \( c_1 = 2 \) and \( c_2 = 3 \)

Characteristic Equation: \( r^2 - 2r - 3 = 0 \)

Roots: \( r = 3 \) and \( r = -1 \)

Thus, the sequence \( \{a_n\} \) is a solution to the recurrence relation IFF

\[ a_n = \alpha_1 3^n + \alpha_2 (-1)^n \]

for some constants \( \alpha_1 \) and \( \alpha_2 \)
From the initial conditions, it follows that:

\[ a_0 = 1 = \alpha_1 + \alpha_2 \]
\[ a_1 = 1 = \alpha_1 (3) + \alpha_2 (-1) \]

Solving these two equations yields: \( \alpha_1 = \frac{1}{2} \) and \( \alpha_2 = \frac{1}{2} \)

Therefore, the solution to the recurrence relation and initial conditions is the sequence \( \{a_n\} \) with:

\[ a_n = \frac{1}{2}(3)^n + \frac{1}{2}(-1)^n \]
Let: \( a_0 = 1, \quad a_1 = -2, \quad \text{and} \quad a_n = 5a_{n-1} - 6a_{n-2} \)

We see that \( c_1 = 5 \) and \( c_2 = -6 \)

**Characteristic Equation:** \( r^2 - 5r + 6 = 0 \)

**Roots:** \( r = 2 \) and \( r = 3 \)

Thus, the sequence \( \{a_n\} \) is a solution to the recurrence relation IFF

\[
a_n = \alpha_1 2^n + \alpha_2 3^n
\]

for some constants \( \alpha_1 \) and \( \alpha_2 \)
From the initial conditions, it follows that:

\[
\begin{align*}
a_0 &= 1 = \alpha_1 + \alpha_2 \\
a_1 &= -2 = \alpha_1 \cdot 2 + \alpha_2 \cdot 3
\end{align*}
\]

Solving these two equations yields:

\[
\begin{align*}
\alpha_1 &= 5 \\ \alpha_2 &= -4
\end{align*}
\]

Therefore, the solution to the recurrence relation and initial conditions is the sequence \( \{a_n\} \) with:

\[
a_n = 5(2)^n - 4(3)^n
\]
Solving Recurrence Relations, Example V

Let: \( a_0 = 0, \quad a_1 = 1, \quad \text{and} \quad a_n = a_{n-1} + 6a_{n-2} \)

We see that \( c_1 = 1 \) and \( c_2 = 6 \)

**Characteristic Equation:** \( r^2 - r - 6 = 0 \)

**Roots:** \( r = 3 \) and \( r = -2 \)

Thus, the sequence \( \{a_n\} \) is a solution to the recurrence relation IFF

\[
a_n = \alpha_1 3^n + \alpha_2 (-2)^n
\]

for some constants \( \alpha_1 \) and \( \alpha_2 \)
Solving Recurrence Relations, Example V

From the initial conditions, it follows that:

\[
\begin{align*}
a_0 &= 0 = \alpha_1 + \alpha_2 \\
a_1 &= 1 = \alpha_1 (3) + \alpha_2 (-2)
\end{align*}
\]

Solving these two equations yields:

\[
\alpha_1 = \frac{1}{5} \quad \text{and} \quad \alpha_2 = -\frac{1}{5}
\]

Therefore, the \textit{solution} to the recurrence relation and initial conditions is the sequence \( \{a_n\} \) with:

\[
a_n = \frac{1}{5}(3)^n - \frac{1}{5}(-2)^n
\]
What To Do When There’s Only One Root?

Theorem 1 does not apply when there is a single characteristic root of multiplicity two.
Theorem 2. Let \( c_1 \) and \( c_2 \) be real numbers. Suppose that

\[
r^2 - c_1 r - c_2 = 0
\]

has only one root, \( r_0 \). Then the sequence \( \{a_n\} \) is a solution of the recurrence relation

\[
a_n = c_1 a_{n-1} + c_2 a_{n-2}
\]

if and only if

\[
a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n
\]

for \( n = 0, 1, 2, \ldots \), where \( \alpha_1 \) and \( \alpha_2 \) are constants.

Notice the **extra factor** of \( n \) in the second term!
Let: \( a_0 = 1, \ a_1 = 6, \text{ and } a_n = 6a_{n-1} - 9a_{n-2} \)

We see that \( c_1 = 6 \) and \( c_2 = -9 \)

**Characteristic Equation:** \( r^2 - 6r + 9 = 0 \)

**Root:** \( r = 3 \) with multiplicity 2

Thus, the sequence \( \{a_n\} \) is a solution to the recurrence relation \( IFF \)

\[
a_n = \alpha_1 3^n + \alpha_2 n(3)^n
\]

for some constants \( \alpha_1 \) and \( \alpha_2 \)
From the initial conditions, it follows that:

\[
\begin{align*}
a_0 &= 1 = \alpha_1 \\
a_1 &= 6 = \alpha_1 (3) + \alpha_2 (3)
\end{align*}
\]

Solving these two equations yields: \( \alpha_1 = 1 \) and \( \alpha_2 = 1 \)

Therefore, the solution to the recurrence relation and initial conditions is the sequence \( \{a_n\} \) with:

\[
a_n = (3)^n + n(3)^n
\]
Single Root, Example II

Let: \( a_0 = 1, \quad a_1 = 3, \quad \text{and} \quad a_n = 4a_{n-1} - 4a_{n-2} \)

We see that \( c_1 = 4 \) and \( c_2 = -4 \)

**Characteristic Equation:** \( r^2 - 4r + 4 = 0 \)

**Root:** \( r = 2 \) with multiplicity 2

Thus, the sequence \( \{a_n\} \) is a solution to the recurrence relation IFF

\[
a_n = \alpha_1 2^n + \alpha_2 n2^n
\]

for some constants \( \alpha_1 \) and \( \alpha_2 \)
From the initial conditions, it follows that:

\[
\begin{align*}
a_0 &= 1 = \alpha_1 \\
a_1 &= 3 = \alpha_1 (2) + \alpha_2 (2)
\end{align*}
\]

Solving these two equations yields: \( \alpha_1 = 1 \) and \( \alpha_2 = \frac{1}{2} \)

Therefore, the solution to the recurrence relation and initial conditions is the sequence \( \{a_n\} \) with:

\[
a_n = 2^n + \frac{1}{2}n2^n = 2^n + n2^{n-1}
\]
Single Root, Example III

Let: \(a_0 = 1, \ a_1 = 12, \ \text{and} \ a_n = 8a_{n-1} - 16a_{n-2}\)

We see that \(c_1 = 8\) and \(c_2 = -16\)

**Characteristic Equation:** \(r^2 - 8r + 16 = 0\)

**Root:** \(r = 4\) with multiplicity 2

Thus, the sequence \(\{a_n\}\) is a solution to the recurrence relation IFF

\[a_n = \alpha_1 4^n + \alpha_2 n4^n\]

for some constants \(\alpha_1\) and \(\alpha_2\)
Single Root, Example III — Cont.

From the initial conditions, it follows that:

\[
\begin{align*}
a_0 &= 1 = \alpha_1 \\
a_1 &= 12 = \alpha_1 (4) + \alpha_2 (4)
\end{align*}
\]

Solving these two equations yields: \( \alpha_1 = 1 \) and \( \alpha_2 = 2 \)

Therefore, the solution to the recurrence relation and initial conditions is the sequence \( \{a_n\} \) with:

\[
a_n = (4)^n + 2n(4)^n
\]
Single Root, Example IV

Let: \( a_0 = 2, \quad a_1 = 5, \) and \( a_n = 2a_{n-1} - a_{n-2} \)

We see that \( c_1 = 2 \) and \( c_2 = -1 \)

**Characteristic Equation:** \( r^2 - 2r + 1 = 0 \)

**Root:** \( r = 1 \) with multiplicity 2

Thus, the sequence \( \{a_n\} \) is a solution to the recurrence relation IFF

\[
a_n = \alpha_1 1^n + \alpha_2 n(1)^n
\]

for some constants \( \alpha_1 \) and \( \alpha_2 \)
From the initial conditions, it follows that:

\[
\begin{align*}
    a_0 &= 2 = \alpha_1 \\
    a_1 &= 5 = \alpha_1 (1) + \alpha_2 (1)
\end{align*}
\]

Solving these two equations yields: \( \alpha_1 = 2 \) and \( \alpha_2 = 3 \)

Therefore, the solution to the recurrence relation and initial conditions is the sequence \( \{a_n\} \) with:

\[
a_n = 2(1)^n + 3n(1)^n = 2 + 3n
\]
Solving Recurrence Relations

**Definition.** A linear homogeneous recurrence relation of degree $k$ with constant coefficients is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$. 
**Theorem 3.** Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \ldots - c_k = 0$$

has $k$ distinct roots, $r_1, r_2, \ldots, r_k$. Then the sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_k r_k^n$$

for $n = 0, 1, 2, \ldots$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.
Multiple Distinct Roots, Example I

Let: \( a_0 = 2, \ a_1 = 5, \ a_2 = 15, \ \text{and} \)
\[
a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}
\]

We see that \( c_1 = 6, \ c_2 = -11, \ \text{and} \ c_3 = 6 \)

**Characteristic Equation:**
\[
r^3 - 6r^2 + 11r - 6 = (r - 1)(r - 2)(r - 3) = 0
\]

**Roots:** \( r = 1, \ r = 2, \ \text{and} \ r = 3 \)

Thus, the sequence \( \{a_n\} \) is a solution to the recurrence relation IFF
\[
a_n = \alpha_1 1^n + \alpha_2 2^n + \alpha_3 3^n
\]

for some constants \( \alpha_1, \ \alpha_2, \ \text{and} \ \alpha_3 \)
From the initial conditions, it follows that:

\[
\begin{align*}
    a_0 &= 2 = \alpha_1 + \alpha_2 + \alpha_3 \\
    a_1 &= 5 = \alpha_1 + \alpha_2 \cdot 2 + \alpha_3 \cdot 3 \\
    a_2 &= 15 = \alpha_1 + \alpha_2 \cdot 4 + \alpha_3 \cdot 9
\end{align*}
\]

Solving: \( \alpha_1 = 1 \), \( \alpha_2 = -1 \), and \( \alpha_3 = 2 \)

Therefore, the solution to the recurrence relation and initial conditions is the sequence \( \{a_n\} \) with:

\[
a_n = 1 - 2^n + 2 \cdot 3^n.
\]
Let: \( a_0 = 4, \ a_1 = -9, \ a_2 = -9, \) and 
\[ a_n = 4a_{n-1} - a_{n-2} - 6a_{n-3} \]

We see that \( c_1 = 4, \ c_2 = -1, \) and \( c_3 = -6 \)

**Characteristic Equation:**
\[ r^3 - 4r^2 + r + 6 = (r + 1)(r - 2)(r - 3) = 0 \]

**Roots:** \( r = -1, \ r = 2, \) and \( r = 3 \)

Thus, the sequence \( \{a_n\} \) is a solution to the recurrence relation IFF
\[ a_n = \alpha_1 (-1)^n + \alpha_2 2^n + \alpha_3 3^n \]
for some constants \( \alpha_1, \ \alpha_2, \) and \( \alpha_3 \)
Multiple Distinct Roots, Example II — Cont.

From the initial conditions, it follows that:

\[ a_0 = 4 = \alpha_1 (-1)^0 + \alpha_2 2^0 + \alpha_3 3^0 \]
\[ = \alpha_1 + \alpha_2 + \alpha_3 \]

\[ a_1 = -9 = \alpha_1 (-1)^1 + \alpha_2 2^1 + \alpha_3 3^1 \]
\[ = -\alpha_1 + 2 \alpha_2 + 3 \alpha_3 \]

\[ a_2 = -9 = \alpha_1 (-1)^2 + \alpha_2 2^2 + \alpha_3 3^2 \]
\[ = \alpha_1 + 4 \alpha_2 + 9 \alpha_3 \]
Multiple Distinct Roots, Example II — Cont.

Solving: \( \alpha_1 = 5 \), \( \alpha_2 = 1 \), and \( \alpha_3 = -2 \)

Therefore, the solution to the recurrence relation and initial conditions is the sequence \( \{a_n\} \) with:

\[
a_n = 5(-1)^n + 2^n - 2(3)^n.
\]
Solutions to General Recurrence Relations

The next theorem states the most general result about linear homogeneous recurrence relations with constant coefficients, allowing the characteristic equation to have **multiple** roots.

**Key Point**: for each root $r$ of the characteristic equation, the general solution has a summand of the form $P(n)r^n$, where $P(n)$ is a polynomial of degree $m - 1$, with $m$ the **multiplicity of this root**.
Theorem 4. Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \ldots - c_k = 0$$

- has $t$ distinct roots, $r_1, r_2, \ldots, r_t$, with
- multiplicities $m_1, m_2, \ldots, m_t$, respectively, so
  - $m_i \geq 1$ for $i = 1, 2, \ldots, t$, and
  - $m_1 + m_2 + \ldots + m_t = k$. 
Then a sequence \( \{a_n\} \) is a solution of the recurrence relation

\[
a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}
\]

if and only if

\[
a_n = (\alpha_{1,0} + \alpha_{1,1}n + \ldots + \alpha_{1,m_1-1}n^{m_1-1})r_1^n \\
+ (\alpha_{2,0} + \alpha_{2,1}n + \ldots + \alpha_{2,m_2-1}n^{m_2-1})r_2^n \\
+ \ldots \\
+ (\alpha_{t,0} + \alpha_{t,1}n + \ldots + \alpha_{t,m_t-1}n^{m_t-1})r_t^n
\]

for \( n = 0, 1, 2, \ldots \), where the \( \alpha_{i,j} \) are constants

for \( 1 \leq i \leq t \) and \( 0 \leq j \leq m^i - 1 \)
Multiple Roots, Example I

If a linear homogeneous recurrence relation has a characteristic equation with roots 2, 2, 2, 5, 5, and 9, then the form of a general solution is:

\[
a_n = (\alpha_{1,0} + \alpha_{1,1} n + \alpha_{1,2} n^2)2^n + (\alpha_{2,0} + \alpha_{2,1} n)5^n + (\alpha_{3,0})9^n
\]
Multiple Roots, Example II

Let: \( a_0 = 1, \ a_1 = -2, \ a_2 = -1, \) and
\[
a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}
\]

We see that \( c_1 = -3, \ c_2 = -3, \) and \( c_3 = -1 \)

**Characteristic Equation:** \( r^3 + 3r^2 + 3r + 1 = 0 \)

Since \( r^3 + 3r^2 + 3r + 1 = (r + 1)^3, \) the characteristic equation has a single root, \( r = -1, \) of multiplicity three.

By Theorem 4., the solutions of this recurrence relation are of the form:
\[
a_n = \alpha_{1,0} (-1)^n + \alpha_{1,1} n(-1)^n + \alpha_{1,2} n^2(-1)^n
\]
for some constants \( \alpha_{1,0}, \ \alpha_{1,1}, \ \text{and} \ \alpha_{1,2} \)
Multiple Roots, Example II — Cont.

From the initial conditions, it follows that:

\[
\begin{align*}
a_0 &= 1 = \alpha_{1,0} (-1)^0 + \alpha_{1,1} 0^1(-1)^0 + \alpha_{1,2} 0^2(-1)^0 \\
a_1 &= -2 = \alpha_{1,0} (-1)^1 + \alpha_{1,1} 1^1(-1)^1 + \alpha_{1,2} 1^2(-1)^1 \\
a_2 &= -1 = \alpha_{1,0} (-1)^2 + \alpha_{1,1} 2^1(-1)^2 + \alpha_{1,2} 2^2(-1)^2 \\
\end{align*}
\]

or

\[
\begin{align*}
1 &= \alpha_{1,0} \\
-2 &= - \alpha_{1,0} - \alpha_{1,1} - \alpha_{1,2} \\
-1 &= \alpha_{1,0} + 2 \alpha_{1,1} + 4 \alpha_{1,2}
\end{align*}
\]
Solving these three equations simultaneously yields:

\[ \alpha_{1,0} = 1, \quad \alpha_{1,1} = 3, \quad \alpha_{1,2} = -2 \]

Thus, the unique solution to the recurrence relation and initial conditions is the sequence \( \{a_n\} \) with:

\[ a_n = (1 + 3n - 2n^2)(-1)^n \]
Let: \( a_0 = 1, \ a_1 = 1, \ a_2 = 2, \) and
\[
a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}
\]

We see that \( c_1 = 3, \ c_2 = -3, \) and \( c_3 = 1 \)

**Characteristic Equation:** \( r^3 - 3r^2 + 3r - 1 = 0 \)

Since \( r^3 - 3r^2 + 3r - 1 = (r - 1)^3 \), the characteristic equation has a single root, \( r = 1 \), of multiplicity three.

By Theorem 4., the solutions of this recurrence relation are of the form:
\[
a_n = \alpha_{1,0} (1)^n + \alpha_{1,1} n(1)^n + \alpha_{1,2} n^2(1)^n
\]

for some constants \( \alpha_{1,0}, \ \alpha_{1,1}, \) and \( \alpha_{1,2} \)
From the initial conditions, it follows that:

\[ a_0 = 1 = \alpha_{1,0} (1)^0 + \alpha_{1,1} 0^1(1)^0 + \alpha_{1,2} 0^2(1)^0 \]
\[ a_1 = 1 = \alpha_{1,0} (1)^1 + \alpha_{1,1} 1^1(1)^1 + \alpha_{1,2} 1^2(1)^1 \]
\[ a_2 = 2 = \alpha_{1,0} (1)^2 + \alpha_{1,1} 2^1(1)^2 + \alpha_{1,2} 2^2(1)^2 \]

or

\[ 1 = \alpha_{1,0} \]
\[ 1 = \alpha_{1,0} + \alpha_{1,1} + \alpha_{1,2} \]
\[ 2 = \alpha_{1,0} + 2 \alpha_{1,1} + 4 \alpha_{1,2} \]

Solving these three equations simultaneously yields:

\[ \alpha_{1,0} = 1, \quad \alpha_{1,1} = -\frac{1}{2}, \quad \alpha_{1,2} = \frac{1}{2} \]
Thus, the unique solution to the recurrence relation and initial conditions is the sequence \( \{ a_n \} \) with:

\[
    a_n = (1 - \frac{1}{2}n + \frac{1}{2}n^2)(1)^n \\
    = 1 - \frac{1}{2}n + \frac{1}{2}n^2 \\
    = \frac{2 - n + n^2}{2}
\]
Let: \( a_0 = 0, \ a_1 = 1, \ a_2 = 2, \ a_3 = 3, \) and 
\( a_n = 2a_{n-2} - a_{n-4} \)

We see that \( c_1 = 0, \ c_2 = 2, \ c_3 = 0, \) and \( c_4 = -1 \)

**Characteristic Equation:**  
\( r^4 - 0r^3 - 2r^2 - 0r + 1 = 0 \)

or,  
\( r^4 - 2r^2 + 1 = 0 \)

Since  
\( r^4 - 2r^2 + 1 = (r^2 - 1)^2 = (r - 1)^2(r + 1)^2, \) the characteristic equation has two roots, \( r_1 = 1 \) and \( r_2 = -1, \) each of multiplicity two.

Solutions of this recurrence relation are of the form:  
\[
a_n = (\alpha_{1,0} + \alpha_{1,1} n)(1)^n + (\alpha_{2,0} + \alpha_{2,1} n)(-1)^n
\]

for some constants \( \alpha_{1,0}, \ \alpha_{1,1}, \ \alpha_{2,0}, \) and \( \alpha_{2,1} \)
Multiple Roots, Example IV — Cont.

From the initial conditions, it follows that:

\[ a_0 = 0 = (\alpha_{1,0} + \alpha_{1,1} 0^1)(1)^0 + (\alpha_{2,0} + \alpha_{2,1} 0^1)(-1)^0 = \alpha_{1,0} + \alpha_{2,0} \]

\[ a_1 = 1 = (\alpha_{1,0} + \alpha_{1,1} 1^1)(1)^1 + (\alpha_{2,0} + \alpha_{2,1} 1^1)(-1)^1 = \alpha_{1,0} + \alpha_{1,1} - \alpha_{2,0} - \alpha_{2,1} \]

\[ a_2 = 2 = (\alpha_{1,0} + \alpha_{1,1} 2^1)(1)^2 + (\alpha_{2,0} + \alpha_{2,1} 2^1)(-1)^2 = \alpha_{1,0} + 2\alpha_{1,1} + \alpha_{2,0} + 2\alpha_{2,1} \]

\[ a_3 = 3 = (\alpha_{1,0} + \alpha_{1,1} 3^1)(1)^3 + (\alpha_{2,0} + \alpha_{2,1} 3^1)(-1)^3 = \alpha_{1,0} + 3\alpha_{1,1} - \alpha_{2,0} - 3\alpha_{2,1} \]

Solving these three equations simultaneously yields:

\[ \alpha_{1,0} = \alpha_{2,0} = \alpha_{2,1} = 0 \text{ and } \alpha_{1,1} = 1 \]
Thus, the unique solution to the recurrence relation and initial conditions is the sequence \( \{a_n\} \) with:

\[
a_n = (0 + 1n)1^n + (0 + 0n)(-1)^n
\]

\[
= n
\]