# Academic Challenge 2012 Regional Math Solutions

1. Ans C: $\dfrac{\binom{8}{4}\binom{11}{2}}{\binom{19}{6}} = \dfrac{(70)(55)}{27132} = .142$

2. Ans C: As the rational expression for f(x) everywhere x is not 3 factors into $\dfrac{(x-2)(x-3)}{x-3}$, it is evident that $f(x) = x - 2$ except at x = 3. Thus, to find the limit, we merely need to plug 3 in for x in the expression x – 2, and we receive 3 – 2 = 1.

3. Ans D: To find the point on the opposite side of the pole, we need to take the opposite sign of the given *r* value (2). The angle, in radians, stays the same. So the correct response is $\left(-2, \frac{2\pi}{3}\right)$.

4. Ans E: There are 30 ways to pick the two denominations for the full house, six ways to pick the denomination that shows up thrice and then only five possible denominations left to pick the denomination that shows up twice. There are then C(5, 3) = 10 ways to pick three of the five dice to show up as the larger denomination. By the multiplication principle, we may find the product of the denomination permutations and dice choices in order to answer our question.

5. Ans E: $A(B+D) = B + C \Rightarrow\ AB + AD = B + C \Rightarrow\ AB - B = C - AD \Rightarrow\ B(A-1) = C - AD \Rightarrow$ $B = \dfrac{C - AD}{A - 1}$

6. Ans A: $80 = \dfrac{n}{2}\left(3 + 3 + (n-1)\frac{2}{3}\right) \Rightarrow 240 = n^2 + 8n \Rightarrow n^2 + 8n - 240 = 0$. Then *n* = 12.

7. Ans B: The distance from home at time t can be given as $d = \sqrt{x^2 + y^2} = \sqrt{(5+4t)^2 + (7+8t)^2}$, which simplifies to $d = \sqrt{80t^2 + 152t + 74}$. If we set that equal to 500 and then square both sides, we get $80t^2 + 152t + 74 = 250000$. If we then subtract 250000 from both sides, we get $80t^2 + 152t - 249926 = 0$. Once we use the quadratic formula, we get $t = \dfrac{-152 \pm \sqrt{152^2 + 4 \bullet 80 \bullet 249926}}{2 \bullet 80}$. The positive root of that is approximately equal to B.

8. Ans C: If the two numbers are x and y, then x + y = 56, xy = 65, y = 56 – x, x(56 – x) = 65, $x^2 - 56x + 65 = 0$. This gives $x = \dfrac{-(-56) \pm \sqrt{(-56)^2 - 4(1)(65)}}{2(1)} = 28 \pm \sqrt{719}$. We find that one of these is x, and the other is y. The larger of the two is $28 + \sqrt{719}$.

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9. Ans B: $4y - 5x \Rightarrow 4(5t+7) - 5(4t+2) \Rightarrow 20t + 28 - 20t - 10 \Rightarrow 18$.

10. Ans A: Based on the fact that i to a power uses the following pattern, i, -1, -i, 1, repeat, this simply boils down to i + i + -1 + 1, which is 2i.

11. Ans E: $16x^2 - 96x = 9y^2 + 36y + 36$ may be written in the form $\frac{(x-3)^2}{9} - \frac{(y+2)^2}{16} = 1$. Then $c = \sqrt{9+16} \Rightarrow c = 5$. Since $e = \frac{c}{a}$ and $a = 3$, we know $e = \frac{5}{3}$.

12. Ans B: Since it is decaying exponentially, we know that $Q(t) = Q_o e^{kt}$. We can also replace some values based on the fact that the initial quantity is 200g and the quantity after 53 years has elapsed is 80g. So an equation would look something like this: $80 = 200e^{53k}$. This tells us that $.4 = e^{53k}$. So $k = \frac{\ln.4}{53}$. Now, if we wanted to see how long it would take for exactly half of the substance to remain, we could set the quantity after time to 100g, and solve the following equation for t: $100 = 200e^{\frac{t\ln.4}{53}}$. If we do that, we see that $.5 = e^{\frac{t\ln.4}{53}}$ and thus $\ln.5 = \frac{t\ln.4}{53}$. Thus, t, our half-life, turns out to be $\frac{53\ln.5}{\ln.4}$.

13. Ans C: If x is the amount of the original paycheck, then $\frac{4}{5} \cdot \frac{1}{2} \cdot \frac{3}{4} x = 600$, so x = 2000.

14. Ans E: The rows of the matrix in question must match the number of columns in the given matrix which makes the number of rows 4. The columns of this matrix must match the number of columns in the matrix that is produced. This makes the number of columns 4. We have a 4 by 4 matrix, a square matrix.

15. Ans D: As the depth decreases at a uniform rate, we may consider the pool to have the same volume as a rectangular pool which has a uniform 8 meter depth (8 meters is the average depth over the pool). So the volume may be found by multiplying together 100, 20 and 8, which is 16,000.

16. Ans D: Think of the trapezoid as three equilateral triangles put together. The perimeter would be five sides of the triangles, so each side is 60 inches. If the side of the triangle is 60 inches, then the height is $30\sqrt{3}$ inches, which is also the height of the trapezoid. The area of the trapezoid would be $\frac{1}{2}(60+120) \cdot 30\sqrt{3}$, or about 4677 square inches

17. Ans B: Using expansion by minors we have $-6(k^2+3) + (2k+30) + 3(2-10k) = 50$. Simplifying we find the quadratic equation $3k^2 + 14k + 16 = 0$. Solving this quadratic provides only one integer value, - 2.

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18. Ans E: Nine dogs and six cats create three furballs (15 divided by 5) per day. That means that three dogs and two cats create a furball a day. Eight dogs and eleven cats create five and a half furballs per day. Let x be the number of furballs a dog creates in a day and y be the number of furballs a cat creates in a day. Then 3x + 2y = 1 and 8x + 11y = 5.5. If we multiply the first equation by 8 and the second by -3, we get 24x + 16y = 8 while -24x – 33y = -16.5. If we add the two equations together, we see that – 17y = -8.5, which eventually becomes y = .5. Then one cat creates half a furball per day, and would create five of them in ten days.

19. Ans C: Both polygons have 4 angles. Therefore, the vertex angle sum in both cases is 360 degrees. Subtracting one from the other leaves a difference of 0.

20. Ans E: As the equation can be rewritten as $y = x - 3 + \frac{8x+3}{x^2+3x+1}$, and we note that the remainder term disappears as we tend toward positive or negative infinity, it becomes obvious that y tends toward x – 3 as we go to the infinities.

21. Ans E: $100\pi r^3 = \frac{4}{3}$, so $r = \sqrt[3]{\frac{75}{\pi}} \approx 2.879$. $SA = 4\pi r^2 = 4\pi \left(\frac{75}{\pi}\right)^{\frac{2}{3}} \approx 104$.

22. Ans B: The area of one-fourth of the circle is given by $\frac{\pi(2)^2}{4} \Rightarrow \pi$. The area of one-fourth of the rhombus is given by $\frac{1}{2}(2)(2) = 2$. Subtract the one-fourth rhombus area from the one-fourth circle area, and we have the area of the shaded region, which is $\pi - 2\,cm^2$.

23. Ans D: As 41 squared is 1681 and 40 squared is 1600, by the Pythagorean Theorem, we get the short leg is 9. The smallest angle is opposite the smaller leg, and adjacent to the longer leg, so the cosine of that angle would be D.

24. Ans A: A+B = 90, A + C = 180, B + C = 180. If you add the left sides together and right sides together, you get 2A + 2B + 2C = 450. This means A + B + C = 225.

25. Ans A: Let $x$ represents the distance between Josie and Tom. Then $\cos(32°) = \frac{x}{11800} \Rightarrow x = 11800\cos(32°) \Rightarrow 10006.9$

26. Ans C: If all of the letters were distinct, there would be 13! possible arrangements thereof. As there are 2 Ks and 2 Es, we must divide 13! by 4 (2 arrangements for the Ks, and 2 arrangements of the 2 Es). As there are 3! = 6 possible arrangements of the 3 Es, we must divide that result by 6 in order to get 259,459,200 possible arrangements of Kennebunkport.

27. Ans D: $\log_8 27 \bullet \log_3 x = 9 \Rightarrow \frac{9}{\frac{\log 27}{\log 8}} = \log_3 x \Rightarrow \frac{9\log 8}{\log 27} = \log_3 x \Rightarrow 3^{9\log_{27} 8} = x$.

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28. Ans A: If we let the width be x, the length of the rectangle would be 8x + 6. So we can create the following equation for the area: $8x^2 + 6x = 97460$. If we subtract 97460 from both sides and use the quadratic formula, we get that the width must be 110 in. and the length must then be 886 in. The perimeter, which is two times the length plus two times the width, must then be 1992 in.

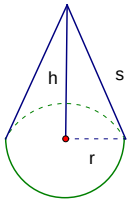
29. Ans B: Of the 100, 100 – 60 = 40 are female, of the blue eyed people, 50 – 40 = 10 are female. This means that of the 40 female, 10, or 25%, have blue eyes.

30. Ans C: $x^{\frac{n}{2}+\frac{m+2}{3}} y^{\frac{m}{2}+\frac{2n}{3}} = x^2 y^3$. Then $\frac{n}{2} + \frac{m+2}{3} = 2$ and $\frac{m}{2} + \frac{2n}{3} = 3$. Using the first equation and solving for n we find $n = \frac{8-2m}{3}$. Substituting this value of n into the second equation and simplifying we find $3m + 4\left(\frac{8-2m}{3}\right) = 18$. Solving for m we find $m = 22$.

31. Ans A: Since $11^2 + 60^2 = 61^2$, the triangle is a right triangle. Since the circle is circumscribed about the triangle, and the large angle is a right angle, the measures of the arcs cut off by the hypotenuse are 180 degrees a piece. That means that the hypotenuse is the diameter of the circle, and its radial length would thus be 30.5 inches. So the area would be pi times 30.5 squared.

32. Ans E: If we treat $\tan^{-1} x$ as the angle y, then $\tan y = x = \frac{x}{1}$. We can set up a right triangle that has y as one of the angles, x as the length of the leg opposite y, and 1 as the length of the leg adjacent to y. This would give the hypotenuse a length of $\sqrt{x^2+1}$. The cosine of y would be adjacent over hypotenuse, or $\frac{1}{\sqrt{x^2+1}}$

33. Ans A: Using the diagram below, let *h* represent the height, *r* represent the radius and *s* represent the slant height:




The area is given by $A = \pi r^2 + \pi rs$. Using Pythagorean theorem and $h = \sqrt{3}\, r$, we can find the slant height as $s = 2r$. Substituting the slant height into the area formula we find $A = \pi r^2 + \pi r(2r) \Rightarrow A = \pi r^2 + 2\pi r^2 \Rightarrow A = 3\pi r^2$.

34. Ans C: There are two possibilities, 4 is the hypotenuse or 4 is not the hypotenuse. (3 cannot be the hypotenuse, as the hypotenuse must be the longest of the three sides.) If 4 is the hypotenuse, by the Pythagorean Theorem, the missing side is the square root of 7. If it's not, then we have a 3-4-5 right triangle.

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35. Ans D: Let *d* represent dimes and *q* represent quarters. Then $4d + 6 = q$. So $(4d + 6) + d = 106$. $d$ must equal 20. The collection consists of 20 dimes and 86 quarters. Finally, $.10(20) + .25(86) = 23.50$

36. Ans D: Let x be the horizontal distance from the 45-degree observer to the airplane. Then the height of the airplane is x feet, as we can make a 45-45-90 triangle between the observer and the plane. With the other observer, we can create a 26-64-90 triangle, and the horizontal distance is 5280 – x feet while the vertical distance is still x feet. So then $\tan 26 = \frac{x}{5280 - x}$. If we solve that for x, we get answer D.

37. Ans D: This population can be modeled by the function $B = 100 \cdot 2^{\left(\frac{t}{5}\right)}$. If we let t = 24, we find that the number of bacteria is 2786. For those who want to use the formula $B = 100 \cdot e^{kt}$, we find that if we solve $2 = e^{k5}$ for k, we get k = $0.2 \ln 2 \approx 0.13863$. Plugging in 24 for t in $B = 100 \cdot e^{0.13863t}$ gives the same result of 2786.

38. Ans E: Writing the given function in factored form, we find $f(x) = \frac{2(x+2)(x-1)}{(x+2)(3x+7)}$. This function has a hole in the graph at x = -2 . After simplifying the function and placing in the value -2 for x we find the corresponding y value is -6. The range cannot contain the value -6. The horizontal asymptote is found by $y = \frac{2x^2}{3x^2} \Rightarrow y = \frac{2}{3}$. We must check to make sure that the graph does not cross this horizontal asymptote. We solve the following equation $\frac{2}{3} = \frac{2(x-1)}{3x+7}$ for x. This gives us $6x + 14 = 6x - 3$ which is a contradiction. The graph does not cross the horizontal asymptote. Therefore y cannot equal $\frac{2}{3}$. We must exclude these two values from our range. $y \in (-\infty, -6) \cup (-6, \frac{2}{3}) \cup (\frac{2}{3}, \infty)$.

39. Ans B: The amplitude of a function that is built from adding a sine function to a cosine function is the square root of the sum of squares of the amplitudes of the sine and cosine functions. Since $11^2 + 48^2 = 2425$, the new amplitude is the square root of that, $5\sqrt{97}$.

40. Ans A: The delays must follow LA < NY < Denver. Chicago can't be the smallest delay, or IV wouldn't work. This means the LA delay is 20 minutes, the New York delay is 30 minutes, the Denver delay is 45 minutes, and the Chicago delay is the 60 minutes. As a quick check, the 60+20 is indeed five minutes longer than the 30+45. This gives a total delay of 20+30+45+60 = 155.