WYSE MATH STATE 2014 SOLUTIONS

1. Ans D: Let R_1 denote residents who participate in the two year program, R_2 denote residents who participate in the four year program, R_3 denote residents who did not participate in crime watch program and V denote house was vandalized.

$P(R_3|V) = \dfrac{P(V|R_3)P(R_3)}{P(V|R_1)P(R_1)+P(V|R_2)P(R_2)+P(V|R_3)P(R_3)}$. We have

$P(R_3|V) = \dfrac{(0.61)(0.33)}{(0.49)(0.24)+(0.28)(0.43)+(0.61)(0.33)} \approx 0.46$

2. Ans C: $\dfrac{d}{dx}\left[\dfrac{e^x}{2}-\dfrac{e^{-x}}{2}\right] = \dfrac{1}{2}e^x + \dfrac{1}{2}e^{-x} = \cosh x$

3. Ans E: There are 2 ways to fill the last position since there are 2 vowels. There are 5 ways to fill the first position since there are 5 consonants. There are 5 letters that are to be arranged between the first consonant and the last vowel. This is done in 5! ways. Then $120 \cdot 5 \cdot 2 = 1200$ ways out of the 7! Total ways. That is $\dfrac{1200}{5040} \Rightarrow \dfrac{5}{21}$.

4. Ans C: $V(L) = L^3$ and $L(t) = 3t$. $V(t) = (3t)^3 = 27t^3$. So $V'(t) = 81t^2$ and in this scenario, t = 6.

5. Ans E: If we treat position as f (x), velocity as f' (x) and acceleration as f" (x), then we start with f" (x) = -2, f' (0) = 60, and f (0) = 0. The antiderivative of f" (x) gives us f' (x) = -2x + C. Since f' (0) = 60, C = 60 and f' (x) = -2x + 60. Taking the antiderivate again gives us $f(x) = -x^2 + 60x + C$. Since f (0) = 0, C = 0 and $f(x) = -x^2 + 60x$. A complete stop implies f' (x) = 0, so -2x + 60 = 0 at x = 30. Evaluating position gives us f (30) = 900.

6. Ans A: There are two highest degree terms that are second degree, namely $7y^2$ and $-14x^2$. Since there is a change of sign, this must be a hyperbola.

7. Ans E: The area under the curve is the answer to A, but that doesn't take into account that the area between the curve and the x-axis from x = -7 to x = 8 would be counted as negative. So we must use the following sum of integrals:

$\int_{-10}^{-7}(x^2 - x - 56)dx + \int_{-7}^{8}(-x^2 + x + 56)dx + \int_{8}^{16}(x^2 - x - 56)dx$

8. Ans D: Using x as years after 2000 and y as population, this should follow the form $y = a \cdot b^x$. We have a = 10000 and b = 11000/10000 = 1.1. This means the population at the start of the year 2015 should be $y = 10000 \cdot 1.1^{15} = 41772$. It is possible to get the same answer by following the form $y = a \cdot e^{kx}$, just be wary to not round k to too few places.

9. Ans E: This is most easily found by taking the magnitude of the cross product between the two vectors. $u \times v = \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 \rangle = \langle -23, 13, 25 \rangle$. The magnitude of that vector would be $\sqrt{(-23)^2 + 13^2 + 25^2} = \sqrt{1323}$.

10. Ans B: $7r = 14 - 7\cos\theta \Rightarrow r = 2 - \cos\theta \Rightarrow r = 2 - \frac{x}{r} \Rightarrow \left(\frac{r^2 + x}{2}\right)^2 = r^2$

$\Rightarrow \frac{r^4 + 2r^2x + x^2}{4} = r^2$. Substituting Cartesian values we have

$\frac{x^4 + 2x^2y^2 + y^4 + 2x(x^2 + y^2) + x^2}{4} = x^2 + y^2$. Simplify and set equal to 0 we have

$x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - 3x^2 - 4y^2 = 0$.

11. Ans C: Since $\det A = -2$, then $-\frac{1}{2}(a_{22}) = \frac{7}{2} \Rightarrow a_{22} = -7$. $-\frac{1}{2}(a_{12}) = -\frac{3}{2}$. So $a_{12} = 3$. However, signs are changed on the minor diagonal when finding the inverse. Therefore $a_{12} = -3$. $-\frac{1}{2}(a_{21}) = 2 \Rightarrow a_{21} = -4$. Again the sign changes on the minor diagonal. So $a_{21} = 4$. We now have $A = \begin{bmatrix} a_{11} & -3 \\ 4 & -7 \end{bmatrix}$. $\det A = -7a_{11} - (-12) \Rightarrow -2 = -7a_{11} + 12 \Rightarrow a_{11} = 2$.

12. Ans E: $\log_x x^3 \cdot \log_y \left(y^2\right)^5 = 3\log_y y^{10} = 3 \times 10 = 30$

13. Ans C: The original function is growing the fastest when the second derivative is 0. The first derivative is $R'(t) = \frac{500e^{-t}}{\left(0.25 + e^{-t}\right)^2}$, the second derivative is $R''(t) = \frac{500e^{-t}(0.25 - e^{-t})}{\left(0.25 + e^{-t}\right)^3}$. The second derivative is 0 when t = -ln(0.25). If we plug this back into the original function, we end up with R = 1000.

14. Ans D: $\cos 3x \cos x + \sin 3x \sin x$ is the general form of $\cos(A - X)$ where $A = 3x$. We have $\cos(3x - x) \Rightarrow \cos(2x)$.

15. Ans E: $\frac{12\cos x}{\sin x} = 5\sin x$. $12\cos x = 5\sin^2 x$. $12\cos x = 5(1 - \cos^2 x) = 5 - 5\cos^2 x$. $5\cos^2 x + 12\cos x - 5 = 0$. So then as the cosine cannot take on a negative value under -1.2, $\cos x = \frac{-12 + \sqrt{144 + 4 \cdot 5 \cdot 5}}{10} = \frac{-12 + \sqrt{244}}{10}$. The arccos of that in degrees would be approximately 69, so if we add 360 degrees to that, we get E.

16. Ans D: If we consider $y = \csc^{-1}(x)$ as the angle of a right triangle, then $\csc y = x/1$. Since cosecant can be thought of as hypotenuse / opposite, we end up with x as hypotenuse, 1 as opposite, and the Pythogorean theorem gives us $\sqrt{x^2-1}$ as adjacent. Since cotangent is adjacent / opposite, we get $\frac{\sqrt{x^2-1}}{1}$.

17. Ans E: Implicit differentiation with respect to x gives us $2x - y - xy' + 2yy' = 0$. Then $(2y - x)y' = y - 2x \Rightarrow y' = \frac{y-2x}{2y-x}$.

18. Ans E: The rule is $\tan^2 x + 1 = \sec^2 x$, so $\tan^2 x - \sec^2 x = -1$ which is always negative.

19. Ans A: $\int \frac{2x+5}{4x}dx \Rightarrow \int \frac{2x+5}{4x}dx \Rightarrow \int \frac{1}{2}dx + \frac{5}{4}\int \frac{1}{x}dx \Rightarrow \frac{1}{2}x + \frac{5}{4}\ln|x| + c$

20. Ans D: There are 6! = 720 possible routes, each of which takes half a minute to check. 360 minutes is 6 hours.

21. Ans E: Without the actual individual values or the sum of squares, the standard deviation cannot be calculated.

22. Ans B: Since angle BAC is bisected, angle BAD and angle CAD are congruent. Similarly angle ACD is bisected so that angle ACB and angle DCB are congruent. Comparing triangle AEC with triangle AEB we find angle EAC congruent to angle EAB, and since BC is a transversal, this makes angles BCD, CBA and EBA congruent to one another. So angle ECA is congruent to angle EBA. Since the angle sum of a triangle is 180 degrees we know that triangle AEC has angles EAC , ACE and CEA that all add up to 180. Similarly triangle AEB has angles EAB, ABE and BEA that also adds to 180. Cancelling out the similar angle measures we find that angles CEA and BEA are congruent. By ASA the two triangles are congruent. Therefore segment AC = AB by cpctc. Also since half of a smaller angle is always less than half of the larger angle, angle DAC must be smaller than angle BCD. Finally, since the measure of angle EAC is less than the measure of angle ACE, it follows that opposite side of angle EAC, namely CE, must be smaller than the opposite side of angle ACE, namely AE.

23. Ans A: At best, it is $\frac{100}{30!}$ percent. And that's ONLY if multiple users don't have the same "most perfect match" – at that point, it's legitimately equal to 0%.

24. Ans A: The left side simplifies to $x + x^2 + x^3 + x^4 + \ldots$. This is a geometric series with a = x and r = x. As long as 0 < x < 1, it will converge to $\frac{x}{1-x}$. The right side is the geometric series $2 + 2x + 2x^2 + 2x^3 + \ldots$ with a = 2 and r = x. Again, as long as 0 < x < 1, it will converges, but this time to $\frac{2}{1-x}$. Solving the equation $\frac{x}{1-x} = 3 - \frac{2}{1-x}$ gives us x = ¼. Since this satisfies the convergence requirement, it is in fact a solution.

25. Ans C: By definition, the angle bisector only cuts the angle in half. The bisector can only cut the opposite side in half only if the angle measures of the triangle are all 60 degrees. Such triangle is called an equilateral triangle.

26. Ans D: f(2) = 3. f(11) = 2047, which has a factor of 23. Such primes are called Mersenne primes, not Gaussian primes. f(4) is 15. f(7) – f(4) = 112.

27. Ans D: Using $BC \cdot CA = CD \cdot CE$ we have $3 \cdot 12 = 2 \cdot CE$. Therefore $CE = 18$. Then $DE = CE - CD \Rightarrow 18 - 2 = 16$.

28. Ans A: Three fourths of two thirds is one half. That means that half of the first twelve years' clutches make it to maturity. The first ten years yield 160 goslings, and the next two yield 16. Half of 176 is 88.

29. Ans C: There must be at least one root for -2, one root for 2, and two roots each for -1 and 1. This makes a minimum of six roots total. It is possible to have more, but a minimum of six are required.

30. Ans E: $3f(x) + 5 = 3x^2 + 2 \Rightarrow 3f(x) = 3x^2 - 3 \Rightarrow f(x) = x^2 - 1$.

31. Ans B: Each inscribed square contains half the area of the prior one. One half raised to the fourth power is one sixteenth.

32. Ans A: If the altitudes are perpendicular to each other and each altitude is by definition perpendicular to their sides, the two sides must be perpendicular to each other. Note: Perpendicular bisectors can't be perpendicular in a triangle. Perpendicular medians and perpendicular angle bisectors can happen, but the resulting triangles are quite varied and have no particular name attached to them.

33. Ans A: $0.20(10{,}000 - x) + x = 5000 \Rightarrow 2000 + 0.80x = 5000 \Rightarrow 0.80x = 3000 \Rightarrow x = 3750$.

34. Ans D: $(2 + \mathrm{i})^4 = 16 + 4 \cdot 8 \cdot \mathrm{i} + 6 \cdot 4 \cdot \mathrm{i}^2 + 4 \cdot 2 \cdot \mathrm{i}^3 + \mathrm{i}^4 = 16 + 32\mathrm{i} - 24 - 8\mathrm{i} + 1 = -7 + 24\mathrm{i}$.
The magnitude is $\sqrt{7^2 + 24^2} = \sqrt{625} = 25$.

35. Ans B: Let $C(x)$ represent the total cost of the group and x represent the size of the group. Then $C(x) = (60 - 0.50(x - 30))x \Rightarrow C(x) = -0.50x^2 + 75x$. The maximum group size is given by $x = -\dfrac{b}{2a}$ where $a = -0.50$ and $b = 75 \Rightarrow x = -\dfrac{75}{2(-0.50)} \Rightarrow x = 75$.
Finally, $C(75) = -0.50(75)^2 + 75(75) \Rightarrow C(75) = 2812.50$.

36. Ans A: $\mathrm{y} = \dfrac{2}{3} \cdot 3\mathrm{t} + 7 = \dfrac{2}{3}\mathrm{x} + 7$

37. Ans B: Certainly the easiest method would be to look at the graph on a graphing calculator. Otherwise, simplify as follows: $\frac{1}{x}+\frac{1}{x+1}-\frac{2}{x-1}\Rightarrow$ $\frac{(x+1)(x-1)+x(x-1)-2x(x+1)}{x(x+1)(x-1)}\Rightarrow \frac{x^2-1+x^2-x-2x^2-2x}{x(x+1)(x-1)}\Rightarrow \frac{-3x-1}{x(x+1)(x-1)}$ which has an x-intercept at x = -1/3.

38. Ans C: Let x represent the number of gallons. Then $25x+40(51-x)=1800\Rightarrow -15x=-240$. Therefore $x=16$. That is 16 gallons for city driving. $25(16)=400$. 400 city miles traveled. There is 35 gallons left with $1625-400=1225$ miles left to travel. Then $\frac{1225}{35}=35$. The average on the highway was 35mi/gal.

39. Ans B: The volume of the cone is $\frac{1}{3}\cdot 2^2\cdot 10\pi=\frac{40\pi}{3}$ cubic inches. The volume of the ice cream is $\frac{1}{2}\cdot 2^3\cdot \pi=4\pi$ cubic inches. So the total volume is $\frac{52\pi}{3}$ cubic inches, approximately 23% of which is taken up by ice cream.

40. Ans B: Erin never had a gold charm and Felicia always included at least one in exchange, so Erin and Felicia must be sisters. Calvin is Anne's brother since he exchanged with Brady and Dana. That means Brady and Dana are brother and sister. Since Calvin didn't end up with a gold charm, Brady and Dana must have them. Technically I and III are superfluous information. Also, be aware that the exchanges are not required to be in simultaneous groups. In theory, the gold charms could be included in every exchange or could be included in as few as two (although that requires a very specific set of circumstances)