WYSE Academic Challenge
Mathematics Test Solutions (Regional) - 2015

1. Answer C: The number of diagonals in an n-gon is $\frac{n(n-3)}{2}$. Thus there are 20 in an octagon and 35 in a decagon.

2. Answer E: When finding the sum or difference of two matrices, both must have the same dimensions or else they cannot be added together.

3. Answer A: ${}_8C_3 \bullet {}_5C_2 \bullet {}_{17}C_1 + {}_8C_3 \bullet {}_5C_3 + {}_8C_4 \bullet {}_5C_2 = 10{,}780$

4. Answer B: $x = 1, y = \sqrt{3}$. Then $r^2 = 1^2 + \sqrt{3}^2 = 4$. So $r = 2$.
$x = r\cos\theta \Rightarrow 1 = 2\cos\theta \Rightarrow \frac{1}{2} = \cos\theta$. So $\theta = \frac{\pi}{3}$. The polar coordinates are $\left(2, \frac{\pi}{3}\right)$. Then $\left(-2, \frac{\pi}{3} + \pi\right) \Rightarrow \left(-2, \frac{4\pi}{3}\right)$ are polar coordinates as well. Also, $\left(2, \frac{\pi}{3} + 2\pi\right) \Rightarrow \left(2, \frac{7\pi}{3}\right)$ are polar coordinates for $\left(1, \sqrt{3}\right)$.

5. Answer B: $i^{504} = 1$ and $i^{507} = -i$, so their sum would be $1 - i$.

6. Answer C: Since the base is a square, $b^2 = 100 \Rightarrow b = 10$. The surface area is given by $A = 2bs + b^2$. We have $360 = 2(10)s + 10^2$. Solving for s we find $s = 13$ Using Pythagorean Theorem to find the height *h* we can solve the following equation for h: $\left(\frac{1}{2}(10)\right)^2 + h^2 = 13^2$. $h = 12$. The volume of the square pyramid is given by $V = \frac{1}{3}b^2h$. $V = \frac{1}{3}(10)^2(12) = 400$.

7. Answer D: Area of the square minus the area of the quarter circle is $2^2 - \pi(2)^2 \div 4$. $4 - 4\pi/4 = 4 - \pi$.

8. Answer E: The length of the base of the parallelogram is found by finding the distance from $(-4, 1)$ and $(4, 1)$. This distance is given by $|4 - (-4)| = 8$. Next, we drop a perpendicular to the base that goes through the point $(-1, 5)$. The perpendicular intersects the base at $(-1, 1)$. We can now find the height of the parallelogram (also the height of the right triangle formed by the perpendicular). The distance from the point $(-1, 5)$ to $(-1, 1)$ is given by $|5 - 1| = 4$. The length of the other side of the right triangle is found using the points $(-4, 1)$ and $(-1, 1)$. This length is 3 units. We now use Pythagorean theorem to find the hypotenuse side. This is the classic 3,4,5 right triangle. So the hypotenuse, the slant height of the parallelogram is 5 units. The perimeter is 2(8) + 2(5) = 26.

9. Answer B: By the binomial theorem,
$(x-2y)^5 = x^5 - 10x^4y + 40x^3y^2 - 80x^2y^3 + 80xy^4 - 32y^5$.
$1-10+40-80+80-32=-1$.

10. Answer E: A regular hexagon is made up of 6 equilateral triangles. Since the perimeter is 60 inches, each side of the hexagon must be 10 inches. Therefore, one side of the equilateral triangle represents the radius of the circle. The radius is 10 inches.

11. Answer A: $\frac{54m^3 - 18m^2n - 6mn^2 + 2n^3}{9m^2 - n^2} = \frac{(3m-n)(18m^2 - 2n^2)}{9m^2 - n^2}$
$= \frac{2(3m-n)(9m^2 - n^2)}{9m^2 - n^2} = 6m - 2n$.

12. Answer B: Let *x* represent the cost of an X-Men shirt and n represent the number of X-Men shirts. Then $xn = 1000$ and
$(x-5)(n+10) = 1000 \Rightarrow xn + 10x - 5n - 50 = 1000$. It follows that
$xn + 10x - 5n - 50 = xn \Rightarrow 10x - 5n - 50 = 0$. Substitute n for $\frac{1000}{x}$ and multiply the equation by $\frac{x}{10}$, we have $x^2 - 5x - 500 = 0$ or $(x-25)(x+20) = 0$ in factored form. Therefore $x = 25$, $x = -20$. In context of this word problem only 25 makes sense. Each shirt cost $25.

13. Answer B: This is a geometric sequence with a common ratio of $\sqrt[14]{\frac{14}{43}}$ and a first term of 43, so the sum of the sequence $\left(s_n = a_1\left(\frac{1-r^n}{1-r}\right)\right)$ would be
$43 \bullet \frac{1}{1-\sqrt[14]{\frac{14}{43}}} = 558.26 \approx 558.$

14. Answer D: There are 11 letters in pyrotechnic, one of which is repeated twice. So the number of distinguishable strings is $\frac{11!}{2!} = 19,958,400$.

15. Answer D: Using D = RT and solving for T, we get that the trip upstream takes $\frac{65}{2.5} = 26$ hours. Since the gondolier loses 2 mph in speed against the current, the current is 2 mph. Thus he rows 6.5 mph with the current, so the return trip takes $\frac{65}{6.5} = 10$ hours. The total trip thus takes 36 hours.

16. Answer C: Since $x^2 + y^2 = 8$, $y^2 = 8 - x^2$. So $3x^2 - \left(8 - x^2\right) = 4 \Rightarrow 4x^2 - 8 = 4$. Solving for x, we find $x = \pm\sqrt{3}$. But x must be greater than 0, so $x = \sqrt{3}$. Plugging in $\sqrt{3}$ for x in the circle equation and solving for y, we find $y = \pm\sqrt{5}$. But *y* must also be greater than 0, so $y = \sqrt{5}$. The solution is $\left(\sqrt{3}, \sqrt{5}\right)$.

17. Answer B: There are 8 vertices in a rectangular prism and $\frac{8(8-1)}{2} = 28$ possible segments involving those vertices. Since 12 of those segments are edges of the figure, there are 16 diagonals.

18. Answer D: $\sqrt[3]{5}\sqrt{2} \Rightarrow 5^{\frac{1}{3}} 2^{\frac{1}{2}} \Rightarrow 5^{\frac{2}{6}} 2^{\frac{3}{6}} \Rightarrow \left(5^2 2^3\right)^{\frac{1}{6}} \Rightarrow 200^{\frac{1}{6}} \Rightarrow \sqrt[6]{200}$

19. Answer E: Since the det is -136, $\begin{vmatrix} 4 & 5 \\ k & 5 \end{vmatrix} - k\begin{vmatrix} -5 & 5 \\ 2 & 5 \end{vmatrix} + 2\begin{vmatrix} -5 & 4 \\ 2 & k \end{vmatrix} = -136$. Then, $20 - 5k - k(-25 - 10) + 2(-5k - 8) = -136 \Rightarrow 20k + 4 = -136 \Rightarrow k = -7$.

20. Answer A: $\log_2 12x^2 - \log_2 3x = 3 \Rightarrow \log_2 \frac{12x^2}{3x} = 3 \Rightarrow 2^3 = 4x \Rightarrow x = 2$.

21. Answer C: The adult mows one-twelfth of the lawn each minute. The kid mows one-forty eighth of the lawn each minute. That means that they mow five-forty eights of the lawn each minute if they work together. $\frac{1}{12}t + \frac{1}{48}t = 1 \Rightarrow 5t = 48$. 48 divided by 5 is 9.6 minutes. Rounding to the nearest minute gives us 10 minutes.

22. Answer E: Many will forget that there were no stipulations placed on the orientation of the square and will get the 25 1 x 1 squares, the 16 2 x 2 squares, the 9 3 x 3 squares, the 4 4 x 4 squares and the 1 5 x 5 square, for a total of 55 squares. But there are a few more—each oriented "diagonally". There are 9 squares with an area of 2 and 1 with an area of 8, for a total of 65 squares.

23. Answer A: Even functions are functions of the form f(x) for which f(-x) is exactly the same as f(x). It is well-established that cos kx and sec kx are both even and that the product of even functions is still even (as is any multiple of an even function), like –cos 4x (which is another way of writing part d). It is also well-established that sin kx is odd, which is categorically not even.

24. Answer B: The three measures of central tendency are the mean, median and mode.

25 Answer E: $4y^2 - x^2 - 4x - 8y = 4 \Rightarrow 4\left(y^2 - 2y + \underline{1}\right) - \left(x^2 + 4x + \underline{4}\right) = 4 + \underline{4} - \underline{4}$. This equation may be rewritten as $\frac{(y-1)^2}{1^2} - \frac{(x+2)^2}{2^2} = 1$. This hyperbola follows the format of $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ which by definition has the transverse axis parallel to the y-axis.

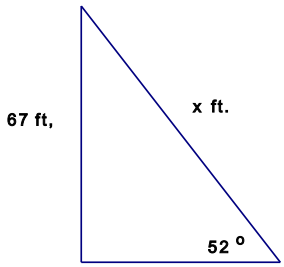
26. Ans C: By setting the denominator equal to 0, we get that there is a vertical asymptote of x = 2. By long division, we get $r(x) = x + \frac{8x-6}{x^3-8}$, so y = x is a slant asymptote. Because the degree of the numerator is one higher than that of the denominator, there is only an oblique asymptote and no horizontal ones, so this closes out the tally of asymptotes at two and no more.

27. Answer C: $r = \sqrt{5^2+12^2} = 13$, $x = 5$, $y = 12$. $\csc\theta = \frac{r}{y} \Rightarrow \csc\theta = \frac{13}{12}$

28. Answer C: $(1-\cos t)^2 + (1-\sin t)^2 = 1 - 2\cos t + \cos^2 t + 1 - 2\sin t + \sin^2 t$. Since the sum of squares of the sine and cosine functions is 1, the answer becomes 3 – 2 cos t – 2 sin t.

29. Answer B: At t = 3, the point is (-3, 17). At t = 10, the point is (-17, 45). In total, the distance traveled would be $\sqrt{(-3+17)^2 + (17-45)^2} = \sqrt{196+784} = \sqrt{980} \approx 31$.

30. Answer D: $x = \frac{67}{\sin(52°)} = 85.024$. Rounds to 85 ft. See diagram below.



31. Answer D: This is an application of the “change of base” formula. The others are all improper use of “rules” which have been creatively applied by my students over the years.

32. Answer B: $p(1,10) = \frac{1}{15}\bullet\frac{1}{14} = \frac{1}{210}$, $p(2,9) = \frac{2}{15}\bullet\frac{2}{14} = \frac{4}{210}$, $p(8,3) = \frac{1}{15}\bullet\frac{1}{14} = \frac{1}{210}$, $p(7,4) = \frac{2}{15}\bullet\frac{3}{14} = \frac{6}{210}$, $p(5,6) = \frac{1}{15}\bullet\frac{1}{14} = \frac{1}{210}$. Adding all the probabilities together we get $\frac{13}{210} = 0.0619 \Rightarrow 6\%$.

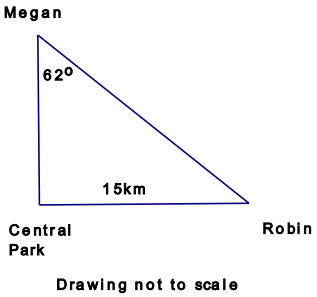
33. Answer B: We can rationalize the denominator of this expression only by multiplying by the expression which would make it a sum of cubes (of cube roots).

34. Answer C: It can't be less than 3 as a team would have to lose a game to score 1 or 2 points, which means that some other team would have at least 3 points. If all teams drew each of their games, there would be a four-way tie with 3 points, and one of those teams would win the group.

35. Answer A: A markup of 38% is the same as multiplying by 1.38. A markdown of 40% is the same as multiplying by 1 - .4 = .6. 1.38 times .6 is .828, which corresponds to a 17.2% markdown.

36. Answer E: The decagon can be cut into ten isosceles triangles by drawing the radii which intersect the decagon at its corners. Then each triangle has a vertex angle of 36 degrees which is included between two radii. So the area of each triangle is $\frac{1}{2}r^2 \sin 36^\circ$, where r stands for the length of the radii. Since there are 10 such triangles, the decagon has an area of $5r^2 \sin 36^\circ$. The circle has an area of πr^2. Then the ratio of the area of the decagon to the area of the circle is $\frac{5\sin 36^\circ}{\pi} \approx 94\%$.

37. Answer A: Using the diagram below, we can find the distance x, from Megan to Robin's house by calculating the following:
$x = 15 / \sin(62°) \Rightarrow x = 16.98855076 \approx 17.0$



38. Answer C: When chords intersect, the two segments of one chord have the same product of lengths as the two segments of the other one. So the three possible known segment pairs would be 4 and 5, 4 and 8 or 5 and 8. If one chord is divided into segments of length 4 and 5, the product would be 20. So 8x = 20 and x = 2.5. If one chord is divided into segments of length 4 and 8, the product would be 32. So 5x = 32 and x = 6.4. If one chord is divided into segments of length 5 and 8, the product would be 40. So 4x = 40 and x = 10.

39. Answer D: Since the ticket cost $1, the number of drawings is the same as the money spent. We solve for x.
$1 - 0.999^x = 0.5 \Rightarrow 0.5 = 0.999^x \Rightarrow \ln 0.5 = x \ln 0.999$
$\Rightarrow \frac{\ln 0.5}{\ln 0.999} = x \Rightarrow x = 692.8005 \approx 693$

40. Answer E: Since Rose's activity takes 1.5 times as long as another (2) she cannot be horseback riding (1). Jane spends time hiking (2), so Rose must be swimming. Let $x \Rightarrow$ the shortest time, $y \Rightarrow$ the middle time and $z \Rightarrow$ the longest time. Then by (1) we have $x + y = z$;
$x + y + z = 200 \Rightarrow x + y + (x + y) = 200 \Rightarrow x + y = 100$. So z = 100. Then by (2) we have $10x = 4z \Rightarrow 10x = 400 \Rightarrow x = 40$. Since $x + y = 100 \Rightarrow 40 + y = 100$, so y must be 60. Rose's time is 1.5 times another time. Since 100 is not 1.5(60), but 60 is 1.5(40), then Rose must have spent 60 minutes swimming.