WYSE – Academic Challenge
Mathematics Test Solutions (Regional) – 2017

1. Ans B: Remember that the sum of the two roots of a quadratic equation in standard form is given by –b/a, which is $\frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$.

2. Ans D: There are two repeated letters in the word archnemesis, each of which is repeated only once each, so we divide 11! by 2! and then 2! again.

3. Ans B: Let r be the radius of the circle, which also serves as the side length of the hexagon. The hexagon is six equilateral triangles of side length r, so the area of the hexagon must be $6 \cdot \frac{1}{2} r \cdot \frac{\sqrt{3}}{2} r = 1$. Solve for r to get r = 0.620403 m. The area of the circle then must be $\pi 0.620403^2 \approx 1.209 \text{ m}^2$

4. Ans A: $i^8 = \left(i^4\right)^2 = 1^2 = 1$.

5. Ans C: Let x be the number of two-legged dogs, y be the number of “tripods”, and z be the number of four-legged dogs. Then 2x + 3y + 4z = 102, x + y + z = 36 and x = 10y. So then 23y + 4z = 102 and 11y + z = 36. Thus 44y + 4z = 144, so 21y = 42 and y = 2. Which means that x must be 20 and the remaining 14 dogs have four legs each.

6. Ans D: By definition, $m\angle A + m\angle B = m\angle C + m\angle D = 180$, and $m\angle B + m\angle C = 90$. $m\angle A + m\angle B + m\angle C + m\angle D = 360 = m\angle A + 90 + m\angle D$, so $m\angle A + m\angle D = 270$.

7. Ans C: Using change of base, $\log_2 x = \frac{\ln x}{\ln 2}$ and $\log_4 x^2 = \frac{\ln x^2}{\ln 4} = \frac{\ln x^2}{2\ln 2}$. Replace these terms to get $\frac{\ln x}{\ln 2} = \frac{\ln x^2}{2\ln 2}$, giving us $2\ln x = \ln x^2$, which is true for all positive values of x.

8. Ans D: Since $Q(t) = 10e^{.041t}$, and Q(t) = 100,000, $10{,}000 = e^{.041t}$ and $.041t = \ln 10{,}000$. So the result is $t = \frac{\ln 10{,}000}{.041}$.

9. Ans E: Let x be the radius of the can, meaning 2x is the height/diameter. The surface area would be $2\pi x^2 + 2\pi x \cdot 2x = 1$. Solve for x to get $\sqrt{\frac{1}{6\pi}} \approx 0.23033$ feet. The volume of the can would be $\pi x^2 \cdot 2x = 2\pi \cdot 0.23033^3 \approx 0.07678$ cubic feet.

10. Ans C: C is the only one of these which is algebraically equivalent and takes into account that $\ln x^8$ has a domain of both positive and negative real numbers.

11. Ans D: There are 7! circular permutations of 8 objects, but if we flip our key rings back and forth, there’s only half as many possible different arrangements.

12. Ans E: Without knowing the lengths of the sides (which could all be different), none of these are guaranteed, especially since there are many possible measures for the fourth angle.

13. Ans A: $\tan^2 x + 1 = \sec^2 x = \dfrac{1}{\cos^2 x} = \left(\dfrac{5}{4}\right)^2$

14. Ans D: The base periods are π, 2π, and 2π respectively. We divide each of those periods by the coefficient of x in order to get the periods, which are then 4, 4, and 32 respectively.

15. Ans C: A regular hexagonal prism has two hexagonal bases and six square sides.

16. Ans E: Using the law of cosines, $b^2 = a^2 + c^2 - 2ac\cos B$, where b is the side opposite point B. So $88^2 = 45^2 + 83^2 - 2 \cdot 45 \cdot 83\cos B$ and $-1{,}170 = -7{,}470\cos B$ and $\cos B = \dfrac{1170}{7470}$.

17. Ans B: Since $y = (x-4)^2 - 5$, $(x-4)^2 = y + 5 = 1(y+5)$. The length of the latus rectum of a parabola which opens upward is the 4p in $(x-h)^2 = 4p(y-k)$.

18. Ans B: $\left(\sqrt{2x+1}\right)^2 = (x-7)^2 \Rightarrow 2x + 1 = x^2 - 14x + 49 \Rightarrow 0 = x^2 - 16x + 48 \Rightarrow$ $0 = (x-12)(x-4) \Rightarrow$ x = 4, 12. However, the 4 is an extraneous solution created by the squaring. Graphing or using a calculator solver would also produce the answer.

19. Ans D: His speed homeward is 125 mph, so his speed in still air would be 137.5 mph. If he has the assistance of the wind, then, his speed would be 150 mph. That means it would take him 20 hours to return, and 24 + 1 + 20 = 45 hours for the full journey. The cookies were outstanding, by the way, and perhaps we'll find a way to work out the recipe in a future installment.

20. Ans E: This fits none of the commonly-given formulas for any of these polar graphs. It instead creates a circle of radius $2\sqrt{2}$ centered on the rectangular point (2, 2).

21. Ans D: Pump A fills 1/40 of the pool per minute, and pump B fills 1/50 of the pool per minute. At 12:05, 5/50 of the pool is full, and running both pumps fills 1/40+1/50 of the pool per minute. Solve for 5/50 + (1/40 + 1/50) x = 1, where x is minutes after 12:05. We end up with x = 20, so approximately 12:25.

22. Ans A: This is twice the classic 1/n series, which is likewise divergent to infinity.

23. Ans D: Let x = calories in a banana and y = calories in an orange. Then we know that 15x + 10y = 1800 and 20x + 8y = 2160. So then 60x + 40y = 7200 and 60x + 24y = 6480, which leaves 16y = 720, y = 45 and x = 90. So 2x + 2y = 270.

24. Ans E: The truck is scheduled to arrive at City B at 200/60 = $3\frac{1}{3}$ hours after noon, which is 3:20. The car is schedule to arrive at City B at 200/75 = $2\frac{2}{3}$ hours after 12:30, which is 3:10. The car will pass the truck. Solve 60T = 75C where T is hours after 12:00

and C is hours after 12:30. Substitute T = C + 0.5, giving us 60(C + 0.5) = 75C, and C = 2 hours after 12:30. The car passes the truck at 2:30.

25. Ans D: As t goes to infinity, $r(t)$ goes to 117 times 4 mph, since e to an increasingly negative power heads toward 0.

26. Ans B: Every potential rational zero takes on the form p/q, where p is an integer factor of the constant term and q is an integer factor of the leading coefficient. Each integer has a factor of plus or minus 1 at a bare minimum, so plus or minus one are the only two **guaranteed possible** rational zeroes of a polynomial with integral coefficients.

27. Ans A: We have three restrictions based on roots and the denominator. $2-x\geq 0$, giving us $x\leq 2$. $x+3\geq 0$, giving us $x\geq -3$. Finally, $1-\sqrt{x+3}\neq 0$, giving us $x\neq -2$. If we put these three restrictions together, we get $[-3,-2)\cup(-2,2]$.

28. Ans A: Since $\cos^2 t+\sin^2 t=1$, $\left(\frac{x}{4}\right)^2+\left(\frac{y}{9}\right)^2=1$ and $\frac{x^2}{16}+\frac{y^2}{81}=1$, which is the equation of an ellipse.

29. Ans A: Singular matrices have a determinant of zero.

$$\begin{vmatrix}3 & 0 & 4\\ 6 & k & 2\\ 7 & -k & 1\end{vmatrix}=3\begin{vmatrix}k & 2\\ -k & 1\end{vmatrix}-0\begin{vmatrix}6 & 2\\ 7 & 1\end{vmatrix}+4\begin{vmatrix}6 & k\\ 7 & -k\end{vmatrix}=3(k+2k)+4(-6k-7k)=3(3k)+4(-13k)=-43k$$

This is only equal to zero if k = 0.

30. Ans A: Since by (I) and (II) the pair of twin girls only ride together on T @ 1, they must each ride W and X separately @ 2. Hence by (III) and (IV), H and E must be the pair of twin girls with H riding X @ 12, T @ 1, and W @ 2 and with E riding W @ 12, T @ 1, and X @ 2. Similarly, by (III) and (II) F must be one of the twin boys and rides X @ 12, T @ 1, and W @ 2. This leaves A and D to each have a twin brother, but by (V) D and G are not siblings. Also by (V), G rides without A, so by (I) A and G are not siblings. Hence, G is the other twin brother with F and rides T @ 12, X @ 1, and W @ 2. Since by (III) and (IV) the grandfather does not ride W or X @ 2, he must ride T @ 2, leaving the only time to ride with G and satisfy (V) is on X @ 1, with D. Thus, the grandfather rides W @ 12, X @ 1, and T @ 2, and D, who by (III) is not on W @ 2, rides W @ 12, X @ 1, and T @ 2. Since by (IV) C rides X @ 2 and B does not, the only remaining ride schedule for C is T @ 12, and W @ 1. Similarly, what remains for B is X @ 12, W @ 1, and T @ 2, and what remains for A is T @ 12, W @ 1, and X @ 2. Reconciling these schedules, we see that B rides with six difference people: F & H @ 12, A & C @ 1, and D & the grandfather @ 2.