WYSE – Academic Challenge
Mathematics Test Solutions (Sectional) – 2017

1. Ans C: For a complex number of the form z = a + bi, any angular argument in polar form, θ, follows the rule $\tan\theta = \frac{b}{a}$. So in this case, any tangent must be equal to $\frac{-5}{3}$. The arctangent of $\frac{-5}{3}$ in degree form is –59.0°. Since the period of the tangent is 180° every argument is a multiple of 180° away from that. The smallest positive argument would be 121.0°.

2. Ans D: For the first two, we use $z = \frac{x-\mu}{\sigma}$, so the z-scores are –2 and 1, respectively. Applying the central limit theorem to the next two, we use the formula $z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}}$, so those particular z-scores are 1 and approximately 1.26, respectively.

3. Ans C: Let x be the length of each side of the squares. The combined area is $2x^2 = 2\text{ m}^2$, giving us x = 1 m. Using the Pythagorean Theorem on a diagonal, the radius of the circle would be $\frac{\sqrt{5}}{2}$ m, and the area of the circle would be $\pi\left(\frac{\sqrt{5}}{2}\right)^2 \approx 3.927\text{ m}^2$.

4. Ans A: This happens to be $f'(1)$. The derivative of f would be 5 times the derivative of $\sqrt{x}$, which is $\frac{1}{2\sqrt{x}}$, so $f'(x) = \frac{5}{2\sqrt{x}}$. Then $f'(1) = 2.5$.

5. Ans D: To find the intersection points, we set f(x) = g(x), giving us $\frac{x}{3} = \sqrt{x}$. Square both sides to get $\frac{x^2}{9} = x$. So $\frac{x^2}{9} - x = 0$ and $x\left(\frac{x}{9} - 1\right) = 0$. Therefore f(x) and g(x) intersect at x = 0 and x = 9. Over this interval, g(x) is the top function on the graph, so the problem is now finding $\int_0^9 \left(\sqrt{x} - \frac{x}{3}\right)dx$. An antiderivative of that integrand would be $\frac{2}{3}x^{3/2} - \frac{x^2}{6}$. Evaluated at 9, we get 4.5. Evaluating that at 0, we get 0, so the answer is 4.5 – 0 = 4.5.

6. Ans E: Population and growth can be modeled by $P = Ae^{kt}$ and $P' = Ake^{kt}$. Starting with t = 0 for the given information, we have A = 100,000 and Ak = 2,000, giving us k = 0.02. The growth ten years from now is $P' = 100000 \cdot 0.02e^{0.02 \cdot 10} \approx 2442.81$ people per year.

7. Ans B: For a conic section in general form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, an acute angle of rotation will satisfy the equation $\cot 2\varphi = \frac{A-C}{B}$, which means in this case that $\cot 2\varphi = \frac{1}{5}$. So $\tan 2\varphi = 5$, which tells us that $\varphi = \frac{\tan^{-1}5}{2} \approx 39.3°$.

8. Ans E: The first four are properties of matrices. The fifth is not, as there are myriad pairs of matrices which multiply one way and not the other, for example. (There are also many more pairs which have one product in one order and a different in the second.)

9. Ans D: Let x be gallons of concentrate added. Then 20 * 0.1 + x * 0.75 = (20 + x) * 0.2, and x = 3.6364.

10. Ans A: It's that time of the year for DeMoivre's Theorem, which states that if a complex number is written in the form z = r ($\cos\theta + i\sin\theta$), then $z^n = r^n\left(\cos(n\theta) + i\sin(n\theta)\right)$. Since r is the modulus of the complex number, z can be written in this particular case as a + bi, where a = 1 and b = –2, $r = \sqrt{a^2 + b^2} = \sqrt{1^2 + (-2)^2} = \sqrt{5}$ and then $r^6 = 125$. $\theta = \tan^{-1}\frac{b}{a}$, so $\theta = \tan^{-1}(-2)$. Then $(1-2i)^6 = 125\left(\cos\left(6\tan^{-1}(-2)\right) + i\sin\left(6\tan^{-1}(-2)\right)\right)$. Alternatively, binomial expansion can be used to solve this.

11. Ans D: We can combine the logarithms using multiplication of arguments, and the result is $\log\left(x^2 + 3x - 990\right) = 3$. Since the log in this case is base 10, we have $x^2 + 3x - 990 = 10^3 = 1000$. Then we have the equation $x^2 + 3x - 1990 = 0$, with solutions $x = \frac{-3 \pm \sqrt{7969}}{2}$ (using the quadratic formula with a = 1, b = 3, and c = –1990). Note that the negative solution is not in the domain of either original logarithm, so it is not a possible solution. The positive one checks.

12. Ans E: Set the length of each edge at x. Using Pythagorean theorem, the diagonals of each triangular face would be $\frac{\sqrt{3}}{2}x$, and the overall height of the pyramid would be $\frac{\sqrt{2}}{2}x$. This means the volume would be $\frac{1}{3}x^2 \cdot \frac{\sqrt{2}}{2}x = \frac{\sqrt{2}}{6}x^3 = 1{,}000$, giving us x = 16.1887. The combined surface area would be $4 \cdot \frac{1}{2}x \cdot \frac{\sqrt{3}}{2}x$, or 453.93 square meters.

13. Ans E: r(x) = x – 5, except when x = –2. Thus its graph is a line with a hole and no asymptotes.

14. Ans C: Let Jack take x hours, so then Justin takes x – 5 hours. Which means that Jack can finish $\frac{1}{x}$ of the fence each hour and Justin can finish $\frac{1}{x-5}$ of the fence each hour. Together, they finish $\frac{1}{4}$ of the fence each hour, so $\frac{1}{x} + \frac{1}{x-5} = \frac{1}{4}$. If we multiply this equation through by the denominators, we get $4(x-5) + 4x = x(x-5)$. Then $8x - 20 = x^2 - 5x$, so $0 = x^2 - 13x + 20$. So $x = \frac{13 \pm \sqrt{89}}{2}$ but the smaller of the two is under 4 hours, so Jack cannot work that fast if the two take 4 hours working in concert.

15. Ans E: By Euclid's theories, A and D are most likely to be parallel with B and C serving as transversals. The information suggests that B and C are not the same line, but it is possible that A and D could be the same, since that would make both A and D perpendicular to both B and C.

16. Ans E: Because these two equations are sinusoidal, x is bounded by –A and A, and y is bounded by –B and B. The rectangle in question has a base of 2A and a height of 2B, so the area is 4AB.

17. Ans E: Clearly the solution needs to have x > 2 and x < 9 in order to be within the domains of both of the logarithms. If we combine the logarithms, we get $\log\frac{x-2}{9-x}<1$. Since 10^x is an ever-increasing function, we can raise 10 to the power of both sides and retain the truth of the inequality (but heed the limitations from before). So $\frac{x-2}{9-x}<10$, and since 9 – x must be positive, x – 2 < 90 – 10x. Thus 11x < 92, so x < 92/11 as well, and the solution set is the open interval (2, 92/11).

18. Ans B: First draw the centers P and Q and lines AP, BP, EP, DQ, BQ, and CQ resulting in two pairs of congruent right triangles. We have $\tan ABP=\frac{40}{80}$, so angle ABP is 26.565 degrees. Angle EBP is also 26.565 degrees, so angles DBQ and CBQ are 63.435 degrees. Now $\tan 63.435=\frac{20}{BC}$, which makes BC 10 inches.

19. Ans C: Let u = 10^x. So u^2 – 8u + 15 = 0. Then (u – 3)(u – 5) = 0, so u = 3 or u = 5. Then 10^x = 3 or 10^x = 5, and thus x = log 3 or log 5.

20. Ans D: 1 revolution is 13π inches. So the unicyclist pedals 520π inches per minute. If we multiply that by 60 minutes in an hour, he pedals 31200π inches per hour. If we divide that by 12 inches per foot, it is now 2600π feet per hour. And if we further divide that by 5,280 feet per mile, he pedals $\frac{65}{132}\pi$ miles per hour.

21. Ans D: A regular icosahedron has twenty triangular faces. This would be 3*20 = 60 total sides of the triangles, but since each edge is a side of two triangles, we get 60/2 = 30.

22. Ans B: The probability of winning on the first try is 0.25 and the payout then is \$32. This contributes \$8 to the expected payout. The subsequent probability of winning on the second try is 0.25, but that needs to be tempered by the fact that there's only a 0.75 chance that we're playing that round. In each round, the probability of winning is that of the prior round, multiplied by 0.75 while the payout needs to be halved. Thus the sum of these expected payouts is a geometric series whose first term is \$8 and whose common ratio is 0.75 * 0.5 = 0.375. Thus the expected payout is $\$\frac{8}{1-.375}=\$\frac{8}{.625}=\$12.80$.

23. Ans D: We would need to divide 8 by 0.5 four times in a row to return to the first term.

24. Ans E: In ascending order, the roots of the numerator and denominator are –A, B, and A. Since none of these roots is repeated, the results of the expression will switch between positive and negative around each point. Plugging in x = 0 gives us $\frac{0-A^2}{0-B}=\frac{A^2}{B}$ which must be positive, so solutions are between –A and B and above A.

25. Ans A: We can use the remainder theorem here. For a polynomial, P(x), P(c) is equal to the remainder when we divide by x – c. When we plug –2 into our particular polynomial, the result is –9.

26. Ans E: Like "Big Brother", when it comes to WYSE, you should "expect the unexpected". If we set the first two sides equal to one another, x = 1, so the dimensions are all length 6. If we do the same with the last two, x = 2, this results in length 9. If we set the first and third equal to one another, x = 4/3, for a length of 23/3. Obviously on a cube, all twelve edges must be equal in length, so there's no way this cube exists.

27. Ans A: Be careful on this one. It takes the hiker 4/2 = 2 hours to climb up the hill and 4/5 = 0.8 hours to go down the hill, for a total time of 2.8 hours to cover the entire 8-mile hike. The average speed is 8/2.8 $\approx$ 2.857 mph, which is not simply the average of the two speeds.

28. Ans D: For trigonometric functions of the form "trig" (k(x – b)), the phase shift is b. We can rewrite these three functions in that factored form to figure out the phase shifts: $\tan\left(2\left(x-\frac{\pi}{12}\right)\right)$, $\sin\left(50\left(x-\frac{2\pi}{25}\right)\right)$, and $\csc\left(75\left(x+\frac{2\pi}{25}\right)\right)$. So both the sine and cosecant functions have the same phase shift (albeit in opposite directions), and those shifts are not *quite* as far as the one from the tangent function.

29. Ans E: The amplitude of this function is given by $5e^{-0.011t}$, so we set that equal to 2. So $e^{-0.011t}=0.4$ and $-0.011t=\ln 0.4$. When we divide both sides by –0.011, we get t is approximately 83.30 seconds.

30. ANS E: We should first notice that Diana must be selling the squash and Bethany must be selling the corn. Eric cannot be selling beans since he has the same number of customers as Bethany. If we let Aaron sell beans, then he must be selling a minimum of 3, which would cause Craig to have 9 customers (too many!). This must mean Craig is selling the beans. Eric cannot be selling the lettuce. If he were, then either Diana would have to sell 1 squash, Eric and Bethany 2 each, leaving 10 for Aaron and Craig (which would make the 3x impossible), or Diana would have to sell 2, making Eric and Bethany sell 4, leaving 5 for Aaron and Craig (also impossible due to the 3x). This means Eric must be selling the tomatoes and Aaron is selling lettuce. Dianna must sell 1 squash, Aaron 2 lettuce, and Craig 6 beans, to get the multiples to work out. This means Bethany sells 3 corn and Eric sells 3 tomatoes.