WYSE – Academic Challenge
Mathematics Test Solutions (State) – 2017

1. Ans C: The distance that it travels downward is given by the series sum expression $\dfrac{20}{1-\frac{1}{3}}$. The upward distance is given by $\dfrac{20/3}{1-\frac{1}{3}}$. These are 30 ft. and 10 ft., respectively.

2. Ans A: The car is travelling 4 miles per minute east and 5 miles per minute north, which is 240 mph east and 300 mph north. The velocity is thus $\sqrt{240^2+300^2}\approx 384.19$ mph.

3. Ans B: Model this situation, use law of cosines based on the angle θ and the rope length L: $L^2=6^2+6^2-2\cdot 6\cdot 6\cdot\cos(\theta(t))$, where L and θ are functions of time t (in seconds). When θ = 45°, L is $6\left(\sqrt{2-\sqrt{2}}\right)$ feet. Next, do an implicit derivative with respect to time to get $2\cdot L\dfrac{dL}{dt}=0+0-2\cdot 6\cdot 6\cdot(-\sin\theta)\cdot\dfrac{d\theta}{dt}$. Plug in the values to get $2\left(6\left(\sqrt{2-\sqrt{2}}\right)\right)0.5=-72\cdot(-\sin(45^\circ))\cdot\dfrac{d\theta}{dt}$, so $\dfrac{d\theta}{dt}=\dfrac{6\left(\sqrt{2-\sqrt{2}}\right)}{36\sqrt{2}}\approx 0.0902\ \dfrac{\text{radians}}{\text{second}}$.

4. Ans D: This equation can be rewritten as $x^2-2x+y^2+6y=-9$. To complete the square, add 1 and 9 to both sides to get $x^2-2x+1+y^2+6y+9=1$, which is equivalent to $(x-1)^2+(y+3)^2=1$, which is the equation of a circle with radius 1 and center (1, –3).

5. Ans B: The cross product is < 4*6 – 5*1, 5*2 – 3*6, 3*1 – 4*2 >, which simplifies to < 19, –8, –5 >. The length of the cross product is the area of the parallelogram determined by those vectors, which is $\sqrt{19^2+(-8)^2+(-5)^2}\approx 21.21$.

6. Ans A: "Growing the fastest" implies a 1st derivative max and a 2nd derivative zero. We get $P'(t)=\dfrac{\left(1+100e^{-0.5t}\right)\cdot 0-10000\left(0+100e^{-0.5t}\cdot(-0.5)\right)}{\left(1+100e^{-0.5t}\right)^2}=\dfrac{500000e^{-0.5t}}{\left(1+100e^{-0.5t}\right)^2}$ and

$$P''(t)=\frac{\left(1+100e^{-0.5t}\right)^2 50000e^{-0.5t}(-0.5)-2\cdot\left(1+100e^{-0.5t}\right)\left(100e^{-0.5t}(-0.5)\right)50000e^{-0.5t}}{\left(\left(1+100e^{-0.5t}\right)^2\right)^2}.$$

As ugly as this is, we can factor and simplify this expression quite a bit. Start with $P''(t)=\dfrac{\left(1+100e^{-0.5t}\right)50000e^{-0.5t}\left(\left(1+100e^{-0.5t}\right)(-0.5)-2\cdot\left(100e^{-0.5t}(-0.5)\right)\right)}{\left(1+100e^{-0.5t}\right)^4}$. Only

$\left(1+100e^{-0.5t}\right)(-0.5)-2\cdot\left(100e^{-0.5t}(-0.5)\right)$ can produce a zero. Simplify, set equal to zero, and solve $-0.5-50e^{-0.5t}+100e^{-0.5t}=0 \Rightarrow 50e^{-0.5t}=0.5 \Rightarrow t=\dfrac{\ln(0.01)}{-0.5}\approx 9.2$ days.

7. Ans A: Since cubic functions all have ranges of all real numbers, they all must have at least one real zero.

8. Ans A: The only restriction is that the radicand must be positive. (It can't be zero, or we're dividing by zero.) Hence, $0<-x^2+6x-8=-\left(x^2-6x+8\right)=-(x-4)(x-2)$. This expression is only positive when one of those linear factors is positive and the other is negative, so (2, 4) is the domain.

9. Ans C: Let x be the radius of the larger circle. Draw all three altitudes of the triangle, effectively splitting it up into six congruent 30-60-90 right triangles. We find that the smaller circle must have a radius of 0.5x, and the complete triangle has a height of 1.5x and a base of $x\sqrt{3}$. This means the respective areas of the large circle, triangle, and small circle are πx^2, $0.75x^2\cdot\sqrt{3}$, and $0.25\pi x^2$. We get the shaded percentage by combining them as follows: $\dfrac{0.75x^2\cdot\sqrt{3}-0.25\pi x^2}{\pi x^2}$. We can cancel out the x's, leaving us with $\dfrac{0.75\cdot\sqrt{3}-0.25\pi}{\pi}\approx 0.163$.

10. Ans A: The second is Euler's constant, the third is pi, the fourth is occasionally referred to as tau, and the fifth is Avogadro's constant (the number of molecules in a mole).

11. Ans B: Since the width of the interval is 3 and there are 3 rectangles, each will have width 1. Since these are left rectangles, their heights will be found by taking e^{x^2+1} with x = –1, 0, and 1, multiplying those heights by 1, and adding them together. This yields $2e^2 + e$, which is approximately 17.496.

12. Ans A: Let x be the common edge length. Each hexagon is six equilateral triangles of side length x, yielding an individual hexagon area of $6\cdot\frac{1}{2}\cdot x\cdot\frac{\sqrt{3}}{2}x$. For twenty hexagons, this is a total area of $30\sqrt{3}\cdot x^2\approx 51.96\cdot x^2$. Each pentagon can be split into five 54-54-72 isosceles triangles with x opposite the 72. The altitude perpendicular to the side length x would be $\frac{1}{2}\cdot x\cdot\tan 54$, yielding an individual pentagon area of $5\cdot\frac{1}{2}\cdot x\cdot\frac{1}{2}x\cdot\tan 54$. For twelve pentagons total, we get a total area of $15\cdot x^2\cdot\tan 54\approx 20.65\cdot x^2$. Combine this with the hexagon total to get $72.61\cdot x^2$. This means a total percentage of black would be 20.65/72.61 ≈ 0.284, or approximately 28%.

13. Ans B: To approximate the change in f, we use the formula $\Delta f\approx f'\left(x_0\right)\cdot\Delta x$ where Δx is equal to the change in input and x_0 is our starting x-value, which are 0.1 and 4, respectively. Since f'(x) = 6x, the change in f is approximately 6 * 4 * 0.1 = 2.4.

14. Ans E: From the left and right, $\lim_{h\to 0^-} \frac{|h|}{h} = -1$ and $\lim_{h\to 0^+} \frac{|h|}{h} = 1$. Since the limit is not the same from both directions, the limit does not exist. Alternatively, the expression looks like a simplified version of the definition of the derivative of |x| at x = 0, which is undefined due to the corner at that value.

15. Ans B: The pyramid will have six triangular faces and the one hexagonal face (it's still a face, even though it's most likely a base on the ground). The fact that the pyramid is skew doesn't change the number of faces. It only changes their relative shape.

16. Ans E: The formula for the amount after 1 yr. would be $Pe^{.1235}$, which is approximately 1.131P, so the annual percentage yield is 13.1%.

17. Ans D: $pH_1 = pH_2 + 2.3$. So $-\log\left[H_1^+\right] = -\log\left[H_2^+\right] + 2.3$. Then $\log\left[H_1^+\right] = \log\left[H_2^+\right] - 2.3$. If we raise 10 to the power of both sides, $10^{\log\left[H_1^+\right]} = 10^{\log\left[H_2^+\right]-2.3}$ and $\left[H_1^+\right] = \left[H_2^+\right]\cdot 10^{-2.3}$.

18. Ans A: The cube has a volume of 10x10x10 = 1000 cubic inches. The sphere has a radius of 6 inches and therefore a volume of $\frac{4}{3}\pi 6^3 \approx 904.8$ cubic inches. The cone has a radius of 7.5 and therefore a volume of $\frac{1}{3}\pi 7.5^2 \cdot 15 \approx 883.57$. This means the cube has the greatest volume, even though the other figures have greater diameters/heights.

19. Ans C: We can use a Poisson distribution function with $\lambda = \frac{13}{2.6} = 5$. (The constant within a Poisson distribution function is the mean number of times the event occurs over a given period of time.) Then $\text{P(10 strikes in 13 seconds)} = \frac{5^{10}e^{-5}}{10!} \approx 0.0181$.

20. Ans C: There are 13 ways to choose the denomination of which we get three, and 4 ways to choose the card of that denomination which is left out. Then there are C(12, 2) = 66 ways to choose the other two denominations and four choices each from the two denominations of which there is only one card. If we multiply those together, we get $13\cdot 4\cdot 66\cdot 4\cdot 4 = 54{,}912$.

21. Ans D: The first faucet adds 1/10 of the pool each hour, the second adds 1/5 each hour, and the drain removes 1/4 each hour. At 2 PM, the pool is down to 2/4 = 1/2 full. From 2 to 4 it loses 1/4 – 1/5 or 1/20 each hour, so by 4 PM it has gone down from 1/2 = 10/20 down to 8/20. From then on, the pool is gaining 1/10 + 1/5 – 1/4 = 1/20 each hour. Since the pool starts at 8/20 at 4 PM, eight hours of filling at 1/20 each hour brings the pool up to 16/20 full, which reduces down to 4/5.

22. Ans B: $2\cos^2\left(\frac{1}{2}z\right) = 2\cos^2\left(\frac{1}{2}z\right) - 1 + 1 = \cos\left(2\cdot\frac{1}{2}z\right) + 1 = \cos z + 1$. Since cos z has a period of 2π, vercos z also has a period of 2π.

23. Ans A: $\cos 2x = 1 - 2\sin^2(x)$. So $\cos 2x - 1 = -2\sin^2(x)$. Then $\frac{1}{2} - \frac{1}{2}\cos 2x = \sin^2(x)$. Since this is a vertically compressed, reflected-over-the-x-axis, and shifted version of cos 2x, whose period is $\frac{2\pi}{2} = \pi$, its period is likewise π.

24. Ans C: Solve to get AD – A = CB – C, A(D – 1) = CB – C, and A = (CB – C)/(D – 1). From the given restrictions, we know D – 1 > 0, so we won't be dividing by zero.

25. Ans D: a and b are both odd functions. The sum and difference of odd functions are also odd. The product and quotient of odd functions are both even.

26. Ans C: $(3 + bi)(3 - di) = 9 - 3di + 3bi - bdi^2 = 9 - 3di + 3bi + bd$

27. Ans B: First calculate some reference points. Kaley starts at the north end at 12:00, reaches the south end 2/5 hours = 24 minutes later (12:24) and returns to the north end twenty-four minutes after that (12:48). Jamel starts at the north end at 12:15, reaches the south end 2/15 hours = 8 minutes later (12:23) and returns to the north 8 minutes after that (12:31). That means there will be two meetings. Jamel will catch up with Kaley just before they both reach the south end. Then he will meet up with her again not long into his return trip north. The first meeting happens when 5K = 15J, that is, when they've reached the same point. Since K's times will be after noon, it's easier to swap out J with J = K – 15/60. Substitute and solve to get $5K = 15(K - 0.25) \Rightarrow 5K = 15K - 3.75 \Rightarrow 3.75 = 10K \Rightarrow K = 0.375$ hours, which is 22.5 minutes after noon. For the meeting on the return, we figure that the two together cover a full four miles, so 5K + 15J = 4. We once again substitute J = K – 0.25, giving us 5K + 15(K – 0.25) = 4. Keep solving this equation to get $5K + 15K - 3.75 = 4 \Rightarrow 20K = 7.75 \Rightarrow K = 0.3875$ hours. This additional amount of time that Kaley ran is 0.3875 – 0.375 = 0.0125 hours. We are asked for the distance, so we multiply this time by Kaley's rate (since her rate determines the distance) to get a final result of 0.0125 * 5 = 0.0625 miles, which rounds to 0.1 miles.

28. Ans B: The determinant of the product of two matrices is the determinant of one times the determinant of the other. The determinant of the first matrix is 35k – 35 and the determinant of the second is 18 – 18 = 0, so the product is 0.

29. Ans E: The probability of getting exactly 5 A's is $\frac{C(30,5)\cdot 4^{25}}{5^{30}}$, as there are C(30, 5) ways to select the 5 A's while the other 25 can be any of the other four answers. Add $\frac{C(30,6)\cdot 4^{24}}{5^{30}}$ and $\frac{C(30,7)\cdot 4^{23}}{5^{30}}$ to that, for a total of approximately 0.506.

30. Ans B: First it is helpful to determine who is related to whom. Since Emily works by herself 10 until noon, she cannot be David's sister. That means Beth is David's sister and Emily is his cousin. Since Andrew always works alone, he cannot be David's father, meaning Carl is David's father. Since Carl can't also be Emily's father, Andrew must be her father. Now let's determine who does what, starting with the last time slot. Beth is gone, Andrew is doing roofing, and Carl and David must be the group doing electrical. This means Emily must be doing flooring. From noon until 2, Emily is gone and David,

Carl, and Beth are the group doing roofing, so Andrew must be doing flooring. From 10 until noon, Andrew does electrical, Emily does roofing, and Beth and David are the group doing siding, so Carl must be the one doing flooring. From 8 until 10, Carl is gone, and Andrew is doing siding. David and Emily are doing plumbing to make sure it gets its two slots, so Beth is doing flooring. Total time : plumbing 2 slots (4 hours), electrical and siding 3 slots (6 hours), flooring 4 slots (8 hours), and roofing 5 slots (10 hours).