WYSE – Academic Challenge
Mathematics Solutions (Regional) – 2018

1. Ans D: Since the denominator is prime and the numerator is not a multiple of it, the fraction is in lowest terms. Since the degree of the numerator is 1 higher than the denominator's degree, it has an oblique asymptote and no horizontal asymptote. Also, the discriminant of the denominator is negative, so it has no real roots, and thus no vertical asymptotes.

2. Ans B: $90° = \frac{1}{2}m\angle ABC + m\angle ABC = \frac{3}{2}m\angle ABC$, therefore $m\angle ABC = 60°$. Then $m\angle ABG = \frac{7}{12}(60°) = 35°$. $m\angle GBC = m\angle ABC - m\angle ABG = 60° - 35° = 25°$.

3. Ans C: There are two repeated letters in the word Desiigner, each of which is repeated only once each, so we divide 9! by 2! and then 2! again to get 90,720.

4. Ans E: $m\angle B = 180 - 100 = 80$, $m\angle A + m\angle B + 60 = 180 \Rightarrow m\angle A + 80 + 60 = 180$, therefore $m\angle A = 40$. Finally, $m\angle A = m\angle x$. So x is 40.

5. Ans A: Because the perimeter is 10 inches, each side is 2 inches. From here, the pentagon can be split up into three isosceles triangles, two of them 36-36-108 triangles, and the middle a 36-72-72 triangle. According to the golden ratio, the two long sides of the middle triangle will have lengths of $1+\sqrt{5}$ inches. We can use Heron's formula to get

$$s_1 = \frac{2+2+(1+\sqrt{5})}{2} \approx 3.6180, \quad A_1 = \sqrt{s_1(s_1-2)(s_1-2)(s_1-(1+\sqrt{5}))} \approx 1.902,$$

$$s_2 = \frac{2+(1+\sqrt{5})+(1+\sqrt{5})}{2} \approx 4.236, \quad A_2 = \sqrt{s_2(s_2-2)(s_2-(1+\sqrt{5}))(s_2-(1+\sqrt{5}))} \approx 3.078,$$

and $A_1 = A_3$, giving a total area of 1.902 + 3.078 + 1.902 = 6.882. Alternatively, we could use the trig formula $A = \frac{1}{2}ab\sin\gamma$ on the same three triangles, or this formula along with law of sines on five equal isosceles triangles. Although all three produce the same result, these two other methods are more vulnerable to rounding error.

6. Ans C: $i^{99} = i^{96}i^3 = \left(i^4\right)^{24} i^3 = i^3 = -i$.

7. Ans B: The triangle's base goes from (0,0) to the intersection of $y = 0$ and $y = -0.5x + k$. In terms of k, this length is $0 = -0.5x + k \Rightarrow x = 2k$. The triangle's height is the y-value of the intersection of $y = 2x$ and $y = -0.5x + k$, at which $2x = -0.5x + k \Rightarrow 2.5x = k$. In terms of k, $x = 0.4k$, giving us $y = 2(0.4k) = 0.8k$. The area is $180 = 0.5(2k)(0.8k)$. $k^2 = 225$, thus $k = 15$.

8. Ans A: It's A. That's the general format of a lemniscate.

9. Ans A: Factor and cancel to get $\dfrac{3x^3 + 19x^2 + 16x - 20}{x^2 + 7x + 10} = \dfrac{(3x-2)(x^2+7x+10)}{x^2+7x+10} = 3x - 2$.

10. Ans D: Cherise, Bill, and the balloon form a 65-108-7 triangle with the 50-foot base opposite the 7-degree angle. Using the Law of Sines, we find that the direct distance from Bill to the balloon is $50\frac{\sin 65}{\sin 7} \approx 371.836$ meters. Bill, the balloon, and the launch pad form a right triangle with $\sin 72 = \frac{h}{371.836}$, so $h = 371.836 \cdot \sin 72 \approx 353.637$ meters. Note that the incorrect answer of 115 m is the distance from Bill to the launch pad.

11. Ans B: For every valid base, the above is true, but the only valid bases of logarithms are positive numbers except for 1.

12. Ans C: If we let x be the number of 1.5-ohm resistors, then 44 – x would be the number of 2-ohm resistors, and $1.5x + 2(44 - x) = 78 \Rightarrow -0.5x = -10 \Rightarrow x = 20$.

13. Ans B: We can write the amount of substance after t hours as $n(t) = n_0\left(\frac{1}{3}\right)^{t/20}$. For 10% of the substance to be left, we have $\frac{1}{10}n_0 = n_0\left(\frac{1}{3}\right)^{t/20}$. If we divide by the initial (nonzero) amount, $\frac{1}{10} = \left(\frac{1}{3}\right)^{t/20}$. If we take an appropriate logarithm of both sides, we get $\log_3 \frac{1}{10} = \frac{t}{20}$ and then $t = 20\log_3 \frac{1}{10} \approx 41.92$ hrs.

14. Ans D: $2x^2 - y^2 = 4(x + y + 1) \Rightarrow \frac{(x-1)^2}{1} - \frac{(y+2)^2}{2} = 1$, which forms a hyperbola.

15. Ans C: The population can be modeled using the function $P = A \cdot 2^{\left(\frac{t}{10}\right)}$. If we let P = 3A, we end up with $3A = A \cdot 2^{\left(\frac{t}{10}\right)}$, $t = 10 \cdot \frac{\ln 3}{\ln 2} \approx 15.8496$. Note that if you use the $2 = e^{k \cdot 10}$ method, rounding during the estimation of k could create a significant amount of error!

16. Ans A: This sort of scenario is exactly why the Poisson distribution was created. Let's all celebrate our success over a plate of tilapia or some other fish.

17. Ans D: $0.40(50) + x = 0.60(x + 50) \Rightarrow 0.4x = 10 \Rightarrow x = 25$, where x represents the ounces of chocolate added.

18. Ans C: The volume of the original cone is $V = \frac{1}{3}\pi r^2 h$. If we let α be the multiplier in the height of the new cone, then $V = \frac{1}{3}\pi(1.6r)^2 \alpha \cdot h$ as well. If we set them equal, we know that 2.56α = 1, so α = 0.390625. The percentage decrease is 100 – 39.0652 = 60.9375%.

19. Ans E: Twice a number cubed implies $2n^3$, then saying less than implies subtracting it off the amount 5n, giving a final result of $5n-2n^3$.

20. Ans B: Drain A empties at a rate of 1/40 of the tank per minute, and drain B empties 1/60 of the tank per minute. If both are running, we have a combined rate of 1/40 + 1/60. To empty the entire tank, we solve (1/40 + 1/60) x = 1 and end up with x = 24 minutes.

21. Ans C: By basic trig identities, $\cot^2 x = \csc^2 x - 1 = \dfrac{1}{\sin^2 x} - 1 = \left(\dfrac{5}{4}\right)^2 - 1 = \dfrac{9}{16}$.

22. Ans D: Start with $V(t) = 10000e^{-0.18t}$. $V(t) = \frac{1}{4} \cdot 10000 = 2500$, solve $2500 = 10000e^{-0.18t}$, giving us the solution $t = \dfrac{\ln \dfrac{2500}{10000}}{-0.18} \approx 7.7016$, which rounds to 7.70 years.

23. Ans E: For y = c + a cos (k(t – b)), the phase shift is b and the period is $\dfrac{2\pi}{k}$. Since the equation can be rewritten as $y = 4 + 3\cos\left(\dfrac{\pi}{4}(t-2)\right)$, we have k = $\dfrac{\pi}{4}$ and b = 2, giving us a phase shift of 8 and a period of 2, making their sum 10.

24. Ans E: This is a lissajous curve with parametric forms $x(t) = a\cos(bt)$; $y(t) = c\sin(dt)$. Since x-values go from –4 to 4, then a = 4. The y-values go from –3 to 3, so c = 3. To find the starting point, we let t = 0 and find the start at the point (4, 0). The curve can be traced by starting in the counter-clockwise direction. The function $x(t)$ starts at and returns to its maximum value once in a period. The function $y(t)$ must have a period half that of $x(t)$ to properly complete the trace of the graph, meaning d = 2b. The only listed parametric equations with these properties are $x(t) = 4\cos t$; $y(t) = 3\sin(2t)$.

25. Ans D: The truck will arrive at City B at 100/40 = 2.5 hours after 9:00, or 11:30, well after the car has left. Solve 40T + 60 C = 100 where T is hours after 9:00 and C is hours after 9:30. Substitute T = C + 0.5, which will give us 40(C + 0.5) + 60C = 100. Solving gives us C = 0.8 hours (48 minutes) after 9:30 AM, which is 10:18 AM.

26. Ans D: The powers of 3 have the sequence of 3, 9, 7, and 1 in the ones place, which then repeats. The corresponding sequence for the powers of 7 is 7, 9, 3, and 1, which also repeats. Thus, these sequences match up on every even-numbered power.

27. Ans E: As per matrix multiplication, $c_{34} = \begin{bmatrix} 1 & k \end{bmatrix} \begin{bmatrix} -2 \\ -4 \end{bmatrix} = -2 - 4k \Rightarrow -10 = -2 - 4k \Rightarrow k = 2$.

28. Ans B: For rational functions with horizontal asymptotes, they tend toward those as x tends toward either of the infinities. Since the one above has the same degree in both the numerator and denominator, its horizontal asymptote is given by the ratio of the leading coefficients of the numerator and denominator, each of which is 1.

29. Ans C: For the geometric series sum formula $S_n = \frac{a_1(1-r^n)}{1-r}$ with r = 0.9 and $a_1 = 30$, consider $n \to \infty$. Since $0.9^n \to 0$ as $n \to \infty$, our expression simplifies to $\frac{30}{1-0.9} = 300$.

30. Ans B: Since D rides with fraternal twins he must be H's identical twin brother. This means D rides R at 2, S at 3, T at 4 (since he can't ride W at 4 with H), and W at 5. H must ride T at 5 (since his brother D isn't riding with F at 2 or 3), W at 4, R at 3, and S at 2. A must also be identical since she rides with fraternal twins, so she must ride R at 5, S at 4 (can't ride with D or H), T at 3, and W at 2. Since A is identical and C has a sister, A and C must be identical twin sisters. This forces C to ride T at 2, W at 3, R at 4, and S at 5 (to avoid riding with identical twins A, D, or H). E rides T at 3 and W at 4 as given, and since she can't ride R at 5 (already taken by A and B), she must ride R at 2 with D, and S at 5 with C. This means E must be B's sister, so this forces F and G to be fraternal twins. At this point, G must ride with D on S at 3, since she's F's sister, and we have our answer. We could finish the remaining unique placement solution, but doing so is not necessary.