WYSE – Academic Challenge
Mathematics Solutions (State) – 2018

1. Ans E: This is an example of "integration by parts", the formula for which is $\int udv = uv - \int vdu$. In this scenario, it is easiest to use u = ln x and $dv = x^2dx$ so $du = \frac{1}{x}dx$ and $v = \frac{x^3}{3}$. This means that $\int vdu = \int \frac{x^3}{3}\frac{1}{x}dx = \int \frac{x^2}{3}dx = \frac{x^3}{9} + C$. So an antiderivative of this particular function (with respect to x) would be of the form $\frac{x^3}{3}\ln x - \frac{x^3}{9} + C$.

2. Ans B: If the floor sits on the xy-plane centered on the origin, the equation for the floor's circle is given by $x^2 + y^2 = 9$ based on a diameter of 6 feet and radius of 3 feet. Assuming the north/south diameter lies along the y-axis, we can determine the distance from the x-axis (which will be used to determine the height of the square cross sections) by solving $x^2 + y^2 = 9$ for y. We get upper and lower bounds of $y = \sqrt{9 - x^2}$ and $y = -\sqrt{9 - x^2}$. The side length of each cross section is $\sqrt{9 - x^2} - \left(-\sqrt{9 - x^2}\right) = 2\sqrt{9 - x^2}$, giving us cross-sectional areas of $\left(2\sqrt{9 - x^2}\right)^2 = 4\left(9 - x^2\right) = 36 - 4x^2$. We calculate the volume using integration: $V = \int_{-3}^{3}\left(36 - 4x^2\right)dx = \left(36x - \frac{4x^3}{3}\right)\Big|_{-3}^{3} = 144$. The lodge contains 144 ft^3.

3. Ans E: Using the double angle identity for tangent, we have $\frac{16\tan x}{1 - \tan^2 x} = 5\tan x$. Thus, tan x = 0 or $\frac{16}{1 - \tan^2 x} = 5$. If tan x = 0, then $x = n\pi$ where n is any integer, giving us answer E. There are no other solutions, because the second equation can be rewritten as $16 = 5 - 5\tan^2 x$. The only way this is possible is if $\tan^2 x = -2.2$, which is impossible.

4. Ans E: Let the horizontal distance between Martha and the kite be represented by x. The vertical distance (constant altitude) is 300 ft. This means x and 300 are the legs of a right triangle, with a hypotenuse of z (the extended amount of string). Using the Pythagorean Theorem, we get the equation $x^2 + 300^2 = z^2$. When z is 500, the length x would be $x = 400$. Note that x and z change while the altitude is held constant at 300 ft. Implicit differentiation gives us $2x\frac{dx}{dt} + 0 = 2z\frac{dz}{dt}$. We are told the horizontal rate of change is $\frac{dx}{dt} = 20$. Substituting in our values $z = 500$, $x = 400$, and $\frac{dx}{dt} = 20$, we have $2(400)(20) = 2(500)\frac{dz}{dt}$. Solving for $\frac{dz}{dt}$ we get $\frac{dz}{dt} = \frac{16000}{1000}$ = 16 ft/sec.

5. Ans A: A regular icosahedron has 20 faces, each a regular triangle. This represents a total of 60 vertices for the triangles, but since each vertex of the icosahedron is shared by 5 triangles, we end up with 60/5 = 12 total vertices for the polyhedron.

6. Ans A: The cross product is $\begin{vmatrix} i & j & k \\ 6 & 9 & 4 \\ 2 & 1 & 8 \end{vmatrix} = i\begin{vmatrix} 9 & 4 \\ 1 & 8 \end{vmatrix} - j\begin{vmatrix} 6 & 4 \\ 2 & 8 \end{vmatrix} + k\begin{vmatrix} 6 & 9 \\ 2 & 1 \end{vmatrix} = 68i - 40j - 12k$.

Adding the coordinates gives us 68 + (-40) + (-12) = 14.

7. Ans D: Since $x = \sec(t)$, then $x^2 = \sec^2(t) = \tan^2(t) + 1$. But since $y^2 = \tan^2(t)$, we can simplify this to $x^2 = y^2 + 1$, which then gives us $x^2 - y^2 = 1$.

8. Ans D: There are 17 balls. There are 9 balls with primary colors (blue, red, yellow). There are C(17, 3) = 680 possible sets of three balls. There are C(9, 3) = 84 ways to get sets of all three primary colors. So then there are 680 – 84 = 596 sets of three balls with at least one non-primary color.

9. Ans C: Note that $A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} v & -t \\ -u & s \end{bmatrix}$ where $\det(A) = sv - tu$. So

$A^{-1} = \begin{bmatrix} \frac{v}{sv-tu} & \frac{-t}{sv-tu} \\ \frac{-u}{sv-tu} & \frac{s}{sv-tu} \end{bmatrix}$, which is equivalent to answer C.

10. Ans E: Let $D(t)$ measure the distance (in feet) the train has travelled t seconds after applying the brakes. Then $D''(t) = -0.5$, $D'(0) = 60$, and $D(0) = 0$. Take the antiderivative of $D''(t)$ to get $D'(t) = -0.5t + C$. Use $D'(0) = 60$ to get C = 60, giving us $D'(t) = -0.5t + 60$. Solve for $D'(t) = 0$ to determine the train stops at t = 120 seconds. Take the antiderivative of $D'(t)$ to get $D(t) = -0.25t^2 + 60t + C$. Use $D(0) = 0$ to determine C = 0, giving us $D(t) = -0.25t^2 + 60t$. Plug in t = 120 to find the train has travelled 3600 feet.

11. Ans B: For $\frac{(x-2)^2}{9} - \frac{(y+5)^2}{16} = 1$, the length of the transverse axis is $2\sqrt{9} = 6$ and the length of the conjugate axis is $2\sqrt{16} = 8$. The difference in lengths is 8 – 6 = 2.

12. Ans B: By rules of logarithms: $3\log(x^2-1) - 2\log(x+1) = \log(x^2-1)^3 - \log(x+1)^2 =$

$\log\frac{(x^2-1)^3}{(x+1)^2} = \log\frac{(x+1)^3(x-1)^3}{(x+1)^2} = \log\frac{(x+1)(x-1)(x-1)^2}{1}$, which may be rewritten as $\log(x^2-1)(x-1)^2$.

13. Ans E: Based on the format $y = A\cos(k(t - B)) + C$, the phase shift is B and the period is $\frac{2\pi}{k}$. Factor the given equation to get $y = -\frac{1}{2}\cos\left(\frac{3}{4}\left(x + \frac{\pi}{3}\right)\right) + 2$. Based on the format, we would have a phase shift of $-\frac{\pi}{3}$ and a period of $\frac{2\pi}{\left(\frac{3}{4}\right)} = \frac{8\pi}{3}$.

14. Ans D: If h is the height of ascent, then $\tan 72° = \frac{h}{100}$, giving $h \approx 307.8$ feet, and a rate of ascent of $\frac{307.8}{10} = 30.78$ feet per second, which rounds to 31 fps.

15. Ans E: The first one translates to $y = 7e^{-2(t+4)} - 6$. The second one translates to $y = 7(e^{-2(t-2)} - 6) + 36$. The third one translates to $y = 7e^{-2(t-2)} - 6$. Both II and III are equivalent to the original function.

16. Ans D: The probability of getting exactly r 6s in n rolls is $C(n,r)\left(\frac{1}{6}\right)^r\left(\frac{5}{6}\right)^{n-r}$. "Three or fewer" out of 20 implies that we should take the sum $\sum_{r=0}^{3} C(20,r)\left(\frac{1}{6}\right)^r\left(\frac{5}{6}\right)^{20-r} \approx 0.56655$.

17. Ans A: Let R_1, R_2, and R_3 represent the event the contestant chooses the prospective routes, and let T represent the event the contestant is trapped. Then using Bayes' Rule, we want $P(R_1|T) = \frac{P(R_1)P(T|R_1)}{P(T)} = \frac{P(R_1)P(T|R_1)}{P(R_1)P(T|R_1) + P(R_2)P(T|R_2) + P(R_3)P(T|R_3)}$ where $P(R_1) = 0.50$, $P(R_2) = 0.25$, $P(R_3) = 0.25$, $P(T|R_1) = 0.80$, $P(T|R_2) = 0.60$, and $P(T|R_3) = 0.30$. Then $P(R_1|T) = \frac{0.50(0.80)}{0.50(0.80) + 0.25(0.60) + 0.25(0.30)} = 0.64$.

18. Ans C: Based on $V = s^3$, each edge has a length of 10 cm. The surface area of a cube is given by $A = 6s^2$. By implicit differentiation, $\frac{dA}{dt} = 6 \cdot 2s \cdot \frac{ds}{dt}$. If we plug in 10 for s and 4 for $\frac{ds}{dt}$, we get $\frac{dA}{dt} = 6 \cdot 2 \cdot 10 \cdot 4 = 480$ cm^2 per second.

19. Ans E: Using similar triangles we can find x, the length of the larger base of the right trapezoid. This means $\frac{120}{140} = \frac{x}{140+84}$ and $x = \frac{120}{140}(224) = 192$. To find the hypotenuse of the large triangle containing both Lots A and B we use the Pythagorean Theorem to get $\sqrt{224^2 + 192^2} \approx 295.03$. The smaller hypotenuse side of Lot A is found the same way: $\sqrt{140^2 + 120^2} \approx 184.39$. The frontage on First Street for Lot B is then the difference in the lengths of these hypotenuses: $295.03 - 184.39 = 110.64$. The sum of the four sides of Lot B is $110.64 + 192 + 84 + 120 = 506.64 \approx 507$.

20. Ans D: Circumference is $2\pi r = 25$, giving us $r = \frac{25}{2\pi}$. Surface area is given by $A = 4\pi r^2$. Then $A = 4\pi\left(\frac{25}{2\pi}\right)^2 \approx 199$.

21. Ans C: Bill completes a single lap every 200 seconds. Fifteen minutes is equal to 900 seconds, so Bill has finished four laps and is 100 seconds (halfway) into a fifth. This gives him a 200-meter head start on Andy. Solving the equation 2t + 200 = 4t gives us a solution of t = 100, or one minute forty seconds. The closest whole minute is 2 minutes after 12:15, which is 12:17.

22. Ans E: If there are x liters of water, then there would be (2,000 – x) liters of 25% saline solution in the mixture. There would also be 0.02(2000) = 40 L of pure saline, so 0.25(2000 – x) = 40. That means however, that 500 – 0.25x = 40. Continue to solve, giving us 0.25x = 460, and x = 1920 L of water. So the chemist needs 80 L of saline, which is more than what is available in the lab.

23. Ans E: There are two groups of solutions, the ones where tan x = 1 and the ones where tan x = –1. The period of tan x is π, and for any real number k, tan x = k has **exactly** one solution per period. This interval has 2 full periods of the tangent function in it, so each of those groups of solutions appears twice. Therefore, there are four solutions in this interval.

24. Ans C: Note that $HA = CA - CH = 14.5 - 2 = 12.5 \Rightarrow 2(12.5) = (DH)^2 \Rightarrow DH = HB = 5$. Therefore $EB = 16$. So $6(16) = 8(EG) \Rightarrow EG = 12$. $GF = EG - EF = 12 - 8 = 4$.

25. Ans C: Let A be the number of flyers Anne took and B be the number Beth took. Anne has 0.30A fliers left and Beth has 0.60B fliers left. Since 0.30A = 0.60B, then A = 2B. Substituting for A gives us 2B + B = 3B original fliers and 0.70(2B) + 0.40B = 1.80B fliers distributed. The fraction of fliers distributed is 1.80B/3B = 0.6, or 60%.

26. Ans D: Because $2^2 = 4$ and $2^5 = 32$, $2^{2x} = 2^{5y}$ and so $2x = 5y$. Thus $x = 2.5y$ which means that x is 250% of y.

27. Ans B: Isolate the $\sqrt{2x-3}$ and square each side to get $\left(\sqrt{2x-3}\right)^2 = \left(2+\sqrt{x+7}\right)^2 \Rightarrow$ $2x-3 = 4+4\sqrt{x+7}+x+7$. Now isolate the $4\sqrt{x+7}$ on the right and square each side to get $(x-14)^2 = \left(4\sqrt{x+7}\right)^2 \Rightarrow x^2-28x+196 = 16(x+7)$. Now simplify, factor, and solve: $x^2-44x+84=0 \Rightarrow (x-2)(x-42)=0 \Rightarrow x=2$ or $x=42$. Now try each of the solutions in the original equation. For x = 2, we have $\sqrt{2\cdot 2-3}-\sqrt{2+7} = 1-3 = -2$, which implies x = 2 is an extraneous root. For x = 42, $\sqrt{2(42)-3}-\sqrt{42+7} = 9-7 = 2$, which implies x = 42 is a true root.

28. Ans D: In the first minute, 200 rabbits each drink one ounce. Using the finite geometric series formula $S_n = \frac{a_1(1-r^n)}{1-r}$ with r = 2 and a_1 = 200, we need to set S_n = 1,280,000. Then $1{,}280{,}000 = \frac{200(1-2^n)}{1-2}$, and upon simplification, we have that –6400 = 1 – 2^n. So 2^n = 6,401. Since the first integer n that satisfies this is n = 13 (which gives us 8,192), it is after 13 minutes when the rabbits have combined for at least 10,000 gallons.

29. Ans A: Let x represent the distance from C to B, giving us $15-x$ as the distance from D to C. Since $AD = 2.5$, use the Pythagorean Theorem, to get $AC = \sqrt{(15-x)^2+(2.5)^2}$. The rowing time can be expressed as $\frac{\sqrt{(x-15)^2+6.25}}{4}$ and the jogging time can be expressed as $\frac{x}{8}$. She completes the task in 3 hours. This gives us the following equation: $\frac{\sqrt{(x-15)^2+6.25}}{4}+\frac{x}{8}=3$. Solving either by graph or by hand gives us a solution of $x \approx 6.74$ (ignoring any solutions that do not satisfy $0 \le x \le 15$). The jogging time given by $\frac{x}{8}$ becomes $\frac{6.74}{8} \approx 0.8425$ hrs. Converting to minutes, we have $60(0.8425) = 50.55 \approx 51$ minutes.

30. Ans C: Since there was only one win/loss record of 3-0 and only one of 0-3, there must be three 2-1 and three 1-2 records. Hal and Fred are definitely 1-2 by I and II, and Erin is 1-2 according to III. This leaves only the 3-0, the three 2-1's, and the 0-3 unaccounted for. To get the right total by IV, the men must be 5-7 overall. Fred and Hal account for 2-4, so Bob and Dave's total is 3-3. Since all 1-2's are accounted for, only the 0-3 and 3-0 are left to give us 3-3. Since we know Bob has lost two games, it leaves only Dave to be the 3-0 victor. (Note that V and VI are technically unnecessary information.)