WYSE – Academic Challenge
Mathematics Solutions (Sectional) – 2019

1. Ans B: Let the height of the pole be h and the distance from the top of the pole to the person whose angle of elevation is 35 degrees be x. The third angle of the overall triangle is $180° - 23° - 35° = 122°$. Using Law of Sines, $\frac{x}{\sin 23°} = \frac{1000}{\sin 122°}$, giving us $x = \sin 23° \frac{1000}{\sin 122°} \approx 460.742$. Since $\sin 35° = \frac{h}{460.742}$, $h = 460.742 \cdot \sin 35° \approx 264.27$.

2. Ans B: Use m as the grams of the 40% chocolate. The amount of 20% chocolate is $200 - m$. This gives us $0.40m + 0.20(200 - m) = 0.35(200) = 70$. Solve to get $m = 150$.

3. Ans A: For any natural numbers k and h such that $h \le k$, C(k, k – h) = C(k, h), the sum of the two values will be k. This can also be shown using symmetry of Pascal's triangle.

4. Ans C: Using D for the number of dimes, N for nickels, and Q for quarters, the information gives the following formulas: $0.05N + 0.10D + 0.25Q = 64$, $D = 3N$, and $Q = 2D - 3$. Substitute to get $Q = 6N - 3$ and $0.05N + 0.10(3N) + 0.25(6N - 3) = 64$, then solve $1.85N = 64.75$, N = 35, $D = 3(35) = 105$, and $Q = 6(35) - 3 = 207$. Total coins is 35 + 105 + 207 = 347.

5. Ans C: We can model the population by the function $P(t) = Ae^{kt}$ and the growth by its derivative $P'(t) = Ake^{kt}$. Based on P(0) = 10,000 and P'(0) = 200, we get A = 10,000 and Ak = 200. This gives us k = 0.02, so $P'(t) = 200e^{0.02t}$ and $P'(20) \approx 298.36$.

6. Ans E: Since x = t + 3, then t = x – 3. This makes $y = (x - 3)^2 + 5(x - 3) - 1 = x^2 - x - 7$.

7. Ans B: Let x be Paul's distance from the intersection, y be Sarah's distance from the intersection, and z be the distance between them. Using the Pythagorean Theorem, we have $z^2 = x^2 + y^2$. When $x = 0.5$ and $y = 1.2$, $z = \sqrt{0.5^2 + 1.2^2} = 1.3$. Differentiating each side of the original equation with respect to t gives us $2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$. Solving for $\frac{dz}{dt}$ gives us $\frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$. Substituting x, y, $\frac{dx}{dt} = -30$, and $\frac{dy}{dt} = -45$ into our $\frac{dz}{dt}$ equation (notice that $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are negative because x and y are decreasing), we find that $\frac{dz}{dt} = -53.0769$. The two are approaching one another at a rate of 53 km/hr.

8. Ans D: There are C(19, 6) = 27,132 possible committees. The number of ways to get an equal number of men and women is C(10, 3) * C(9, 3) = 10,080. We can find the number of ways to get an unequal number of men and women by subtracting these numbers, which gives us 17,052 committees.

9. Ans B: First note that GM = GB – MB = 6, and ΔLGM is similar to ΔWBM. By the Law of Similar Triangles, $\frac{LM}{6} = \frac{12}{9} \Rightarrow LM = \frac{12}{9} \cdot 6 = 8$.

10. Ans C: Completing the square to get: $4\left(x^2+4x\right)+9\left(y^2-2y\right)+16=0 \Rightarrow$

$4\left(x^2+4x+4-4\right)+9\left(y^2-2y+1-1\right)+16=0 \Rightarrow$

$4\left(x^2+4x+4\right)-16+9\left(y^2-2y+1\right)-9+16=0 \Rightarrow$ $4\left(x+2\right)^2+9\left(y-1\right)^2=9 \Rightarrow$

$\frac{\left(x+2\right)^2}{\frac{9}{4}}+\frac{\left(y-1\right)^2}{1}=1 \Rightarrow$ $\frac{\left(x+2\right)^2}{\left(\frac{3}{2}\right)^2}+\frac{\left(y-1\right)^2}{1^2}=1$. The semi-major axis length is thus 1.5, the semi-minor axis length is 1, and the semi-focal length is $\sqrt{1.5^2-1^2}\approx 1.12$. The full focal length is thus 2*1.12 = 2.24.

11. Ans D: The dot product of $\langle a,b,c\rangle$ and $\langle x,y,z\rangle$ is ax + by + cz. In order for two vectors to be orthogonal, their dot product must be 0. All of the given vectors except $\langle 4,-1,4\rangle$ are orthogonal to $\langle 5,6,1\rangle$.

12. Ans E: To return to due south, the total angle of the left turns must be a multiple of $360°$. The shortest third turn must therefore be $360°-90°-120°=150°$.

13. Ans B: If we cross-multiply, (x – 4)(2x + 5) = (x + 2)(x – 5), so $2x^2 - 3x - 20 = x^2 - 3x - 10$. Then $x^2 = 10$, and x = $\pm\sqrt{10}$.

14. Ans A: Alternative 1: Fred 1st, Carrie doesn't place: $\frac{P\left(1,1\right)\cdot P\left(6,2\right)}{P\left(8,3\right)}=\frac{30}{336}$. Alternative 2: Carrie 2nd, Fred doesn't place: $\frac{P\left(6,1\right)\cdot P\left(1,1\right)\cdot P\left(5,1\right)}{P\left(8,3\right)}=\frac{30}{336}$. Add to get $\frac{60}{336}\approx 0.18$.

15. Ans A: The diameter of the circle would be $\frac{1}{\pi}\approx 0.318$. This diameter is the same length as one side of the square, so the square's area would be $\left(\frac{1}{\pi}\right)^2\approx 0.101$.

16. Ans B: $3\cdot\left(7^x\right)^2-7\cdot 7^x+2=0$ and then $\left(3\cdot 7^x-1\right)\left(7^x-2\right)=0$. So $7^x=\frac{1}{3}$ or 2. Then by taking the logarithm base 7 of both sides, $x=\log_7\frac{1}{3}\approx -0.5646$ or $\log_7 2\approx 0.3562$. Add these together to get –0.2084, which rounds to –0.21.

17. Ans C: Let M, N, and O represent the locations of Mary, Nancy, and Oliver respectively. Using Law of Cosines, $MN=\sqrt{60^2+50^2-2\left(60\right)\left(50\right)\cos\left(120°\right)}\approx 95$, where the measure of angle MON is 180° – (27° + 33°) = 120°. It is also possible to do this using two right triangles.

18. Ans A: Clearly their sum is 2x – 6, which must be real if x is real. We can write the components as (x – 3) + i and (x – 3) – i. This clearly marks them as complex conjugates, and their product would be $(x-3)^2-(x-3)i+(x-3)i-i^2 = (x-3)^2+1$, which would be real.

19. Ans E: The total balance would be $A=15000\left(1+\frac{0.06}{24}\right)^{24(1.5)}=16410.77$ using the fact that 18 months equals 1.5 years. The interest would be 16410.77 – 15,000 = 1410.77.

20. Ans D: Let x be the original paycheck. The amount left after rent is x – 0.25x = 0.75x. The amount left after other bills is 0.75x – 0.4*0.75x = 0.6*0.75x = 0.45x. The amount left after food is 0.45x – 0.20*0.45x = 0.8*0.45x = 0.36x = 180. Solve for x to get $500.

21. Ans E: Moving the graph up 3 gives us y = |x| + 3. Moving it left 6 gives us $y=|x+6|+3$. If we vertically stretch by a factor of 9, we must multiply the entire output by 9, giving us $y=9(|x+6|+3)=9|x+6|+27$. Finally, reflecting the graph over the y-axis only negates the x, giving us $y=9|-x+6|+27$.

22. Ans C: If A and B are inverses, then $AB=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$. This means $\begin{bmatrix}1 & k\end{bmatrix}\begin{bmatrix}-3\\ 1\end{bmatrix}=0$, $1(-3)+1k=0$, and $k=3$.

23. Ans E: The rate of change is the derivative $y'=2e^{2t}-16t+\frac{5}{2\sqrt{t}}$, which is undefined when t = 0.

24. Ans A: $\sum_{n=1}^{\infty}\frac{6}{2n^2+6n} = \sum_{n=1}^{\infty}\frac{3}{n(n+3)} = \sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+3}\right) = \left(1-\frac{1}{4}\right)+\left(\frac{1}{2}-\frac{1}{5}\right)+\left(\frac{1}{3}-\frac{1}{6}\right)+\cdots =$
$1+\frac{1}{2}+\frac{1}{3}+\left(\frac{1}{4}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{5}\right)+\cdots = 1+\frac{1}{2}+\frac{1}{3} = \frac{11}{6}$.

25. Ans C: Pump A increases the amount by 1/40 tank per minute. Pump B increases the amount by 1/30 tank per minute. The drain decreases the amount by 1/20 tank per minute. We start with the tank at full, or 1 tank. After 10 minutes of running the drain alone, the tank is down to 1 – 1/20*10 = 1/2 tank (net loss). After 10 minutes more of running the drain and pump A, we get 1/2 – 1/20*10 + 1/40*10 = 1/4 tank left (net loss). Once we turn on pump B, the change per minute is 1/40 + 1/30 – 1/20 = 1/120 tank per minute (net gain). This means eventually the tank will fill. So the time it will take to fill the missing 3/4 of a tank back up will be (3/4) / (1/120) = 90 minutes after 1:20, or 2:50.

26. Ans E: The first one only works for real values when x > 4, but the domain of the original also includes the real interval where x < –1. The same is true for the second statement, which isn't an identity anyway. The third one is the change of base identity.

27. Ans D: The polar radius is $r=\sqrt{(-2)^2+(2)^2}=2\sqrt{2}$. For the polar angle, $\tan(\theta)=\frac{2}{-2}=-1$, giving us $\tan^{-1}(-1)$ or $\pi+\tan^{-1}(-1)$. Since we need to be in the second quadrant, use $\pi+\tan^{-1}(-1)=\frac{3\pi}{4}$. This gives a polar coordinate of $\left(2\sqrt{2}, \frac{3\pi}{4}\right)$.

28. Ans E: The period of sin 3x is $\frac{2\pi}{3}$, and the period of cos 6x is $\frac{2\pi}{6}=\frac{\pi}{3}$. Since the second completes two cycles for each cycle of the first, the overall period is $\frac{2\pi}{3}$.

29. Ans E: Since the length of the parallelogram is the same as the base of the right triangle, the triangle's hypotenuse is $\sqrt{72^2+36^2}\approx 80.498$ feet. The shorter sides of the trapezoid and parallelogram are all $\sqrt{15^2+36^2}=39$ feet. The length of the bottom base of the trapezoid is also 72 feet, and the top base is $72-2(15)=42$ feet. The sum of all sides of the perimeter is thus 80.498 + 36 + 39 + 39 + 42 + 39 + 39 = 314.498 feet. Rounding to a measure divisible by 5, we have 315 feet. After adding the extra 20 feet, the desired total is 335 feet which ends up being $335\div 5=67$ sections of border.

30. Ans A. Number spots 1 (first) to 7 (last). We first conclude Eric must be in spot 7 because of the following reasons: (1) none of the girls can be there, (2) Greg just left that spot, (3) David is between two people, and (4) Brad must be by Felicia to be next to at least one unrelated girl, and Felicia can't be in spot 6. This leaves only two possible boys for spot 1: Brad and Greg. If Greg were in spot 1, then Amy, Brad, and Cindy would have to be taking up spots 2, 4, and 6. This would contradict the fact that David is between two unrelated people. This means Brad must be in spot 1. This puts Felicia in spot 2. Of the four remaining spots, David cannot be in spot 4, since he was there before. David cannot be in spot 6 because Eric must be by a girl. David cannot be in spot 5, since that would not leave a legitimate spot for Cindy (spots 1, 2, and 7 are already taken, so she cannot return to spot 3, and she cannot be in spots 4 or 6 since that would put her back by David). This means David is in spot 3 between Felicia and the middle spot. Since David is between unrelated people and cannot be next to Cindy, the only person available to fill the middle spot is Amy, which finishes our solution. For completion, Greg cannot be in spot 6 (Eric must be by a girl), putting him in spot 5. That puts Cindy in spot 6.