WYSE – Academic Challenge
Mathematics Solutions (State) – 2019

1. Ans D: By the Pythagorean Theorem, the height x of the top of the ladder at the start would be determined by $x^2 + 15^2 = 25^2$, so x = 20. Pushing the ladder increases x to 22. The original angle that the ladder made with the ground was $\sin^{-1}\left(\frac{20}{25}\right) \approx 53.13^\circ$. The new angle that the ladder makes with the ground is $\sin^{-1}\left(\frac{22}{25}\right) \approx 61.64^\circ$. The difference between the two angles is therefore $61.64^\circ - 53.13^\circ = 8.51^\circ$.

2. Ans A: First, reduce by factoring out 2 in the top and bottom, giving us $\frac{2x^3 - 3x^2 + 8x - 2}{x^2 - x + 2}$. Next perform the division

$$\begin{array}{r} 2x - 1 \\ x^2 - x + 2 \overline{\smash{\big)}\, 2x^3 - 3x^2 + 8x - 2} \\ -\left(2x^3 - 2x^2 + 4x\right) \\ \hline -x^2 + 4x - 2 \\ -\left(-x^2 + 1x - 2\right) \\ \hline 3x + 0 \end{array}.$$

The oblique asymptote is the quotient y = 2x – 1.

3. Ans A: The focus is on a line segment that runs perpendicular to the directrix and whose distance from the directrix is twice its distance from the vertex. Since the two lines are perpendicular, their slopes are opposite reciprocal. Since the focus is on the y-axis, the segment containing the focus and the vertex has a y-intercept of 7. So $y = -\frac{1}{2}x + 7$ is an equation for that segment. That line intersects y = 2x when $2x = -\frac{1}{2}x + 7$, which is when $4x = -x + 14$ and x = 2.8. Therefore, y = 2(2.8) = 5.6. The midpoint between (0, 7) and (2.8, 5.6) is the vertex: (1.4, 6.3). By the distance formula, the distance from the focus to the vertex is $\sqrt{(1.4-0)^2 + (6.3-7)^2} \approx 1.57$ units.

4. Ans D: $3\cos\left(\frac{x}{2}\right) = 3 + 3\cos x \Rightarrow \cos\left(\frac{x}{2}\right) = 1 + \cos(x)$. Using the half-angle formula, we have $\pm\sqrt{\frac{1+\cos x}{2}} = 1 + \cos x$. Squaring both sides gives us, $\frac{1+\cos x}{2} = 1 + 2\cos x + \cos^2 x \Rightarrow$ $1 + \cos x = 2 + 4\cos x + 2\cos^2 x \Rightarrow (2\cos x + 1)(\cos x + 1) = 0.$ This means $\cos x$ is either –1 or –0.5. The solutions where $0 \le x \le 2\pi.$ are $x = \frac{2\pi}{3}$, $x = \frac{4\pi}{3}$, and $x = \pi$. But $\frac{4\pi}{3}$ is an extraneous root because it makes the original equation become $-\frac{1}{2} = \frac{1}{2}$ when evaluated at $x = \frac{4\pi}{3}$. The only solutions are $x = \frac{2\pi}{3}$ and $x = \pi$.

5. Ans C: Using integration techniques, $F(x) = 3t - t^2\Big|_0^x = 3x - x^2$, which is positive when $0 < x < 3$.

6. Ans C: Use l'Hospital's Rule to give us $\lim\limits_{x\to 0^-}\dfrac{1-e^{-x}}{x} = \lim\limits_{x\to 0^-}\dfrac{\frac{d}{dx}\left(1-e^{-x}\right)}{\frac{d}{dx}x} = \lim\limits_{x\to 0^-}\dfrac{e^{x}}{1}$, which would be 1.

7. Ans C: A right triangle with legs 5 ft and 12 ft will have a hypotenuse of $\sqrt{5^2+12^2}=13$ feet. Then the area of the top base is $\frac{1}{2}(5\cdot 12)=30$, the total lateral surface area is equal to either of the bases' perimeter times the height, or (5 + 12 + 13)*10 = 300. The total area is therefore 330 ft^2.

8. Ans D: If we let u = $3x^2$ + 8x + 1, then du = (6x + 8)dx and we can thus replace f(x) with any scalar multiple of 6x + 8, including 6x + 8 and 3x + 4.

9. Ans E: $4^x-3\cdot 4^{-x}=2\Rightarrow 4^{2x}-3=2\cdot 4^x\Rightarrow 4^{2x}-2\left(4^x\right)-3=0$. Let $U=4^x$. Then $U^2-2U-3=0$ Then $U=3$ or $U=-1$. So $4^x=3$ or $4^x=-1$, which is impossible. Therefore $4^x=3$ is the only possibility. Solving for x we find $x\log 4=\log 3\Rightarrow x=\dfrac{\log 3}{\log 4}$.

10. Ans D: Let $x=m\angle A$, $y=m\angle B$, and $z=m\angle C$, then x + y = 90, y + z = 180, and 4x = z. Substitute and solve to get y = 90 – x = 180 – 4x, giving x = 30, y = 60, and z = 120.

11. Ans B: The given situation is best modeled by a geometric distribution.

12. Ans B: For a randomly selected cat, let F be the event it's feral, H the event it's a house cat, S be the event it's from a shelter, and T be the event it has the genetic trait. Based on these, we have the following: $P(F)=\dfrac{1}{3}$, $P(H)=\dfrac{5}{12}$, and $P(S)=1-\dfrac{1}{3}-\dfrac{5}{12}=\dfrac{1}{4}$. The conditional probabilities are $P(T|F)=\dfrac{2}{3}$, $P(T|H)=\dfrac{3}{4}$, and $P(T'|S)=\dfrac{3}{5}$, which gives $P(T|S)=\dfrac{2}{5}$. By Baye's Rule, $P(H|T) = \dfrac{P(H)P(T|H)}{P(F)P(T|F)+P(H)P(T|H)+P(S)P(T|S)} =$

$$\frac{\frac{5}{12}\cdot\frac{3}{4}}{\frac{1}{3}\cdot\frac{2}{3}+\frac{5}{12}\cdot\frac{3}{4}+\frac{1}{4}\cdot\frac{2}{5}} = \frac{\frac{5}{16}}{\frac{2}{9}+\frac{5}{16}+\frac{1}{10}}\approx 0.492.$$

13. Ans C: $k(x)=\cos^4 x-\cos 2x = \left(\cos^2 x\right)^2-(2\cos^2 x-1) = \left(\cos^2 x\right)^2-2\cos^2 x+1 = \left(\cos^2 x-1\right)^2$. The inside $\cos^2 x-1$ has a range of values of –1 to 0, but squaring them makes the range of values go from 0 to 1.

14. Ans A: First, $\begin{bmatrix}0 & K & M\end{bmatrix}\begin{bmatrix}K \\ M \\ 1\end{bmatrix}=0 \Rightarrow 0K+KM+M=0 \Rightarrow M(K+1)=0 \Rightarrow$ M = 0 or K = –1.

Second, $\begin{bmatrix}4 & 11 & M\end{bmatrix}\begin{bmatrix}3 \\ 1 \\ -6K\end{bmatrix}=35 \Rightarrow 12+11-6KM=35 \Rightarrow KM=-2$, which means neither K nor M can be equal to 0. Since $K=-1$, we have $(-1)M=-2 \Rightarrow M=2$.

15. Ans D: Surface area $4\pi r^2=100$ means $r=\dfrac{5}{\sqrt{\pi}}$, and volume is $V=\dfrac{4}{3}\pi\left(\dfrac{5}{\sqrt{\pi}}\right)^3 \approx 94.03$.

16. Ans C: The Rational Zeroes Theorem does apply, so the first statement must be true. The graph of a degree three polynomial must pass through the x-axis, so the second statement is true. Depending on the values of a, b, c, and d (such as all equaling 2), plugging in $x=-d$ can result in values other than zero, making the third statement not necessarily true. The polynomial can have two complex zeroes and one real zero, making the last statement also not necessarily true.

17. Ans B: Using the trigonometric identity $\sin 2\theta = 2\sin\theta\cos\theta$, $r=6(2\sin\theta\cos\theta)=12\sin\theta\cos\theta$, $r^3=12(r\cdot\sin\theta)(r\cdot\cos\theta)$. Substituting $r=\sqrt{x^2+y^2}$, $x=r\cdot\cos\theta$, and $y=r\cdot\sin\theta$, we have $\left(\sqrt{x^2+y^2}\right)^3=12xy \Rightarrow \left(x^2+y^2\right)^{3/2}=12xy$, which would be equivalent to $\left(\left(x^2+y^2\right)^{3/2}\right)^2=(12xy)^2 \Rightarrow \left(x^2+y^2\right)^3=144x^2y$.

18. Ans E: $\log(x-5)^2-\log(x-4)^2=\log\dfrac{x^2-10x+25}{x^2-8x+16}=1$. By rewriting in exponential form, $\dfrac{x^2-10x+25}{x^2-8x+16}=10$. So $x^2-10x+25=10x^2-80x+160$ and $9x^2-70x+135=0$. By the quadratic formula, $x=\dfrac{70\pm\sqrt{40}}{18}$. While we could simplify that, we notice that neither solution will be greater than 5. Therefore, there are no real solutions to the original equation.

19. Ans C: The binomial expansion is given by $(a+b)^n=\binom{n}{0}a^nb^0+\binom{n}{1}a^{n-1}b^1+\cdots+\binom{n}{n}a^0b^n$.

The fourth term is $\binom{12}{3}x^9 2^3 = 220\cdot x^9\cdot 8=1760x^9$.

20. Ans B: In terms of meters per minute, Carol's rate is 15*1000/60 = 250, and Dale's rate is 12*1000/60 = 200. Note that Carol will run a total of 250*40/400 = 25 laps and Dale will run a total of 200*30/400 = 15 laps. At 12:10, Dale is at the finish line ready to head counterclockwise, and Carol has run 250*10 = 2500 meters, so she's run 6 laps plus 100 meters, leaving her 300 meters to go clockwise. The two of them will first meet when 250t + 200t = 300, or t = 6/11 minutes after 12:10. From there, they meet every time the two of them have traveled a combined 400 meters, or 250t + 200t = 400, which means t = 8/11 minutes. If we subtract 6/11 from 30 we get 224/11, which is the number of minutes between their first high five and 12:40. If we divide that by 8/11 we get an even 28 (confirming that they do meet at 12:40). That means they meet the first time plus 28 more times, for a total of 29 high fives.

21. Ans B: The product is ac + adi + bci – bd = (ac – bd) + (ad + bc)i. To be real, ad + bc must be equal to 0.

22. Ans E: Solving the given equations for $\sec t$ and $\tan t$, we have $\sec t = \frac{x+3}{2}$ and $\tan t = \frac{y-2}{3}$. Using the identity $1+\tan^2 t = \sec^2 t$, first rearrange to get $\sec^2 t - \tan^2 t = 1$, then substitute to get $\left(\frac{x+3}{2}\right)^2 - \left(\frac{y-2}{3}\right)^2 = 1 \Rightarrow \frac{(x+3)^2}{4} - \frac{(y-2)^2}{9} = 1$.

23. Ans B: The scalar triple product of the vectors is $\begin{vmatrix} 3 & 2 & 3 \\ 6 & 1 & 7 \\ 4 & 0 & 4 \end{vmatrix} = 3\begin{vmatrix} 1 & 7 \\ 0 & 4 \end{vmatrix} - 2\begin{vmatrix} 6 & 7 \\ 4 & 4 \end{vmatrix} + 3\begin{vmatrix} 6 & 1 \\ 4 & 0 \end{vmatrix}$

= $3(4-0) - 2(24-28) + 3(0-4) = 8$.

24. Ans C: Horizontal shift is $-\frac{6\pi}{2} = -3\pi$, therefore, left 3π. Vertical shift is the "– 3", so down 3. The period is $\frac{\pi}{b}$, where b is the coefficient of x. Here $b = 2$, so the period $\frac{\pi}{2}$.

25. Ans B: Solving for x gives us: $x + m = (mx+1)(x-1) \Rightarrow x + m = mx^2 - mx + x - 1 \Rightarrow$ $mx^2 - mx - 1 - m = 0 \Rightarrow x = \frac{m \pm \sqrt{m^2 - 4m(-1-m)}}{2m}$. We get exactly one real solution when $m^2 - 4m(-1-m)$ equals 0, so $m^2 - 4m(-1-m) = 0 \Rightarrow m^2 + 4m^2 + 4m = 0 \Rightarrow$ $5m^2 + 4m = 0$, which has solutions at m = –0.8 and m = 0. If we double check these in the original equation, m = 0 actually ends up giving us an equation with no solution for x, which seems reasonable since it would create division by 0 in the quadratic formula. If we try m = –0.8, we end up with the single real solution of x = 0.5.

26. Ans C: Removing the M and one of the S's leaves us with 12 letters to fill the remaining middle spots. There are 12! not necessarily distinguishable ways to rearrange the middle 12 letters. Since there are three pairs of repeated letters (D, I, P) within these remaining 12 letters, we must divide by 2! to the third power (so divide by 8). 12!/(2!2!2!) = 59,875,200.

27. Ans B: We first note that as the water level drops, the ratio of the radius of the water surface to its depth remains constant due to similar triangles. The radius r of the water surface in terms of its depth h will therefore be equal to $\frac{60}{5}h$. The volume V of the water is then $V = \frac{1}{3}\pi r^2 h$. Substituting 12h for r gives us $V = \frac{\pi}{3}(12h)^2 h = 48\pi h^3$. Using the chain rule we find $\frac{dV}{dt} = 48\pi \cdot 3h^2 \cdot \frac{dh}{dt}$. Given $\frac{dV}{dt} = 65$ and $h = 3$, we find $\frac{dh}{dt} = \frac{65}{48\pi \cdot 27} \approx 0.016$.

28. Ans E: Matt can get a third of a test done per hour, and his cat can undo a fourth of a test per hour. Thus Matt gets 1/3 – 1/4 = 1/12 of a test more done per hour than the cat can undo, so it will take him 12 hours to write a test.

29. Ans C: This cubic function crosses the x-axis at –3, 0, and 2. This function lies above the x-axis from –3 to 0 and lies below the x-axis from 0 to 2. Then the enclosed area that lies between the two curves can be found by adding up the following two integrals:

$A = \int_{-3}^{0}\left(\frac{x^3}{2} + \frac{x^2}{2} - 3x\right)dx + \int_{0}^{2}\left(0 - \left(\frac{x^3}{2} + \frac{x^2}{2} - 3x\right)\right)dx = \frac{253}{24} \approx 10.542$. The nearest whole is 11.

30. Ans B: Assign variables for ages: A for Andrew, B for Beth, C for Claire, D for Danielle, E for Eric, F for Claire's father, J for Uncle Joe, and M for Claire's mother. By II, J is a multiple of 20 (to be divisible by both 4 and 5). By IV we have C=2(D+E), by II we have J=5C, and since D and E can't both be equal to 1, J cannot be as low as 20. If we let J=40, then B=10 and C=8 (by II), D=3 (by VI since J+B–7=J+D), and E=1 (by IV). By VI, we have M+E+C+D+11=M+E+A+B, or C+D+11=A+B, giving us A=12. This gives us F=36 (by III) and M=34 (by V), which means Claire's father is older by 2 years. If we try similar steps with J=60, we get B=15, C=12, and D=8, which by C=2(D+E) would give us E=–2. Trying larger values of J produces similar negative results for E.