WYSE – Academic Challenge
Mathematics Solutions (Regional) – 2021

1. Ans E:If we let n, d, and q be the number of nickels, dimes, and quarters respectively, then $n + d + q = 80$. Sine there are twice as many quarters as nickels, q = 2n. By substitution, $n + d + 2n = 80$, and $d = 80 - 3n$. The value equation is $0.05n + 0.10d + 0.25q = 13$. Substitute again to get $0.05n + 0.10(80 - 3n) + 0.25(2n) = 13$. Solve this to get $n = 20$. This means that $d = 80 - 3(20) = 20$ and $q = 2(20) = 40$.

2. Ans C: There are 12 ways to give the first-place trophy, 11 to give the second-place trophy, and then 10 to give the third-place trophy. The product of those is 1,320.

3. Ans A: The cars will approach each other at a relative speed of 120 mph because one car travels 50 mph and the other travels opposite direction at 70 mph (50 + 70 = 120). They are 240 miles apart, so it takes 240/120 = 2 hours for them to meet. The falcon travels at 80 mph for 2 hours, so the falcon travels 2*80 = 160 miles.

4. Ans E: The quadratic takes on the form a(x – 3 + i)(x – 3 – i), but since a can be anything, the constant can be anything.

5. Ans C: Tom paints 1/10 of a square foot per second, and Bill paints 1/2 of a square foot per second. In the ten-minute, or 600-second, head start, Tom paints 0.1*600 or 60 square feet, leaving 540 square feet to be painted when Tom joins in at 8:10. If we let x be the number of seconds after 8:10, then solving 0.1x + 0.5x = 540 gives us the time the two paint together, which is 900 seconds, or 15 minutes, after 8:10, which is 8:25.

6. Ans E: If B is Bob's current age and S is Sue's current age, $(B-5) = 2(S-5) \Rightarrow B = 2S - 5$. Then $(2S-5) - S = 25 \Rightarrow S = 30$. Therefore, $B = 30 + 25 = 55$. Bob is 55 years old.

7. Ans E: The areas go from s^2 to s^4, so $\frac{s^4 - s^2}{s^2} = s^2 - 1$ represents the relative amount of increase. Since s can be arbitrarily close to 0, any change greater than –100% is acceptable. Since s can be arbitrarily large, any positive percentage is acceptable. Since s can be 1, 0% is also acceptable.

8. Ans C: Since the height is 2 ft, the radius for the semicircle on each side is 1 ft. We need to find the perimeter. $P = 2\pi r + 2(L) = 2\pi(1) + 2(4) = 2\pi + 8$.

9. Ans D: Regular nonagons have equal angles of measure $\frac{(9-2)180^\circ}{9} = \frac{1260^\circ}{9} = 140^\circ$. So we can form 9 isosceles triangles whose base angles are 70 degrees and whose vertex angle is thus 40 degrees. The apothem will be an altitude to the side. We can thus form a right triangle whose small leg is half of the side length and whose other leg is the apothem 8 (which is opposite a 70 degree angle). So $\frac{x/2}{\sin 20^\circ} = \frac{8}{\sin 70^\circ}$ by the Law of Sines. Each side will then be $\frac{16\sin 20^\circ}{\sin 70^\circ}$ So the area of each of the aforementioned isosceles triangles will be $\frac{1}{2}\cdot 8\cdot\frac{16\sin 20^\circ}{\sin 70^\circ} = \frac{64\sin 20^\circ}{\sin 70^\circ}$, and the overall area will be $\frac{576\sin 20^\circ}{\sin 70^\circ} \approx 209.6$.

10. Ans D: Let x be the original monthly paycheck. The rent is 0.2x, leaving x – 0.2x = 0.8x after rent. The bills are 0.4*0.8x, or 0.32x, leaving 0.8x – 0.32x = 0.48x after bills. The food is 0.25*0.48x, or 0.12x, leaving 0.48x – 0.12x = 0.36x left after food. Solving 0.36x=900 gives us \$2500 as the original paycheck.

11. Ans C If we let x and y be the length and width of the garden, the objective function is $A = xy$ with the constraint $6x + 2x + 2(2y) = 8x + 4y = 320$. From the last equation we derive $y = 80 - 2x$. Substituting this value into the area equation, we have $A = x(80-2x) = 80x - 2x^2$. To find the maximum value, either use a graph or $x = \frac{-80}{2(-2)} = 20$. Then y = 80 – 2(20) = 40. The dimensions are 20 ft by 40 ft.

12. Ans D: It's six equilateral triangles with side length 2. Using Heron's formula, each triangle has a semiperimeter of $\frac{1}{2}(2+2+2) = 3$ and area of $\sqrt{3(3-2)(3-2)(3-2)} = \sqrt{3}$. The overall area of the hexagon is thus $6\sqrt{3} \approx 10.4$.

13. Ans D: Long division results in:

$$\begin{array}{r} 2x-1 \\ x^2-x+2\overline{)\,2x^3-3x^2+8x-2} \\ \underline{-\left(2x^3-2x^2+4x\right)} \\ -1x^2+4x-2 \\ \underline{-(-1x^2+x-2)} \\ 3x \end{array}$$

14. Ans C: By the Binomial Theorem, the r^{th} term of $(a+b)^n$ is $C(n,r-1)\cdot a^{n-r+1}\cdot b^{r-1} = C(13,6)\cdot x^7\cdot 2^6 = 1716\cdot 64\cdot x^7 = 109824x^7$.

15. Ans D: We want to know when A decays down to 0.5A with 10% loss a year. Solving the equation $0.5A = A * 0.9^t$ gives us t = ln(0.5)/ln(0.9), or about t = 6.5788 years.

16. Ans C: The volume is $\frac{1}{3}\pi r^2 h$. The base radius is $\frac{1}{2}(23.8) = 11.9$. Next, we use Pythagorean theorem to find the height, $h = \sqrt{s^2 - r^2} = \sqrt{(15.8)^2 - (11.9)^2} \approx 10.39$. The volume is thus approximately $V = \frac{1}{3}\pi(11.9)^2(10.39) \approx 1541$.

17. Ans D: The determinant of a matrix's transpose is the same as that of the matrix. The determinant of the product of two square matrices is the product of their determinants. The product of a scalar and a matrix has a determinant of the scalar times that of the matrix. So, the product of 2, 7, and 5 is 70.

18. Ans D: Our exponential takes the form $f(x) = a(b)^x$. Here a = 1000 , b = 1+ 0.09 = 1.09. The function is $f(x) = 1000(1.09)^x$ where x represents the number of years after 2010. 2021 represents 11 years past 2010. $f(11) = 1000(1.09)^{11} \approx 2580$.

19. Ans E: The answer is $\frac{2\pi}{2/9} = 9\pi \approx 28.3$ as the period of sin kt is pi divided by k.

20. Ans E: The distance value of the polar coordinate is $r = \sqrt{(-4)^2 + (6)^2} \approx 7.211$. Although we traditionally find the angle component using $\theta = \tan^{-1}\left(\frac{6}{-4}\right) \approx -0.983$, we need to adjust this fourth quadrant result into the correct second quadrant by adding pi, so $\theta \approx 2.159$.

21. Ans A: First substitute $t = x + 2$, giving us $y = (x+2)^2 - 4(x+2)$, and then simplify to get $y = x^2 + 4x + 4 - 4x - 8 = x^2 - 4$. This is a parabola with vertex at (0, –4).

22. Ans B: The domain of a logarithm is where its argument is greater than zero. The argument's graph is a parabola opening upward with a vertex of (1, 2). So the argument is always positive, which means that the logarithm can always be evaluated.

23. Ans D: Based on the form $a + ar + ar^2 + ar^3 + \ldots$, we have $a = 8$ and $r = \frac{4}{8} = \frac{1}{2}$. Series of this form have a sum $S = \frac{a}{1-r} = \frac{8}{1-\frac{1}{2}} = 16$.

24. Ans C: Based on $200 \cdot 2^{t/5} = 10000$, solve to get $2^{t/5} = 50 \Rightarrow \frac{t}{5} = \log_2 50 \Rightarrow$ $t = 5\log_2 50 = \frac{5\ln 50}{\ln 2} \approx 28.219$.

25. Ans B: Using an angle of 20 degrees, distance from the wall x and a distance down to the center of the wall of 6.5 feet, solving $\tan\left(20^\circ\right) = \frac{6.5}{x}$ gives us the correct distance of approximately x = 17.859 feet.

26. Ans B: First, complete the square: $9(x^2 - 4x + \underline{4}) + 16(y^2 + 6y + \underline{9}) = -36 + 9 \cdot 4 + 16 \cdot 9$. This gives us $9(x-2)^2 + 16(y+3)^2 = 144$, and $\frac{(x-2)^2}{16} + \frac{(y+3)^2}{9} = 1$. This gives us $a = 4$, so the length of the major axis is 2a = 2(4) = 8.

27. Ans D: Rewrite as $a^{19} - a = 0$, and $a\left(a^{18} - 1\right) = a\left(\left(a^6\right)^3 - 1^3\right) = a\left(a^6 - 1\right)\left(a^{12} + a^6 + 1\right) = a\left(a^3 - 1\right)\left(a^3 + 1\right)\left(\left(a^6\right)^2 + a^6 + 1\right) = 0$. The first factor gives us a = 0, the second a = 1, the third a = –1, and the fourth is quadratic in form with a negative discriminant, so it has no real solutions.

28. Ans A: Using the second row of A and the column of B being set equal to the middle element of the product, $\begin{bmatrix} k & -4 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -5 \end{bmatrix} \Rightarrow 4k - 12 - 1 = -5 \Rightarrow k = 2.$

29. Ans A: For rational functions where the denominator has a higher degree, the horizontal asymptote is y = 0.

30. Ans B: Let us consider the flute player and determine her name. We know she can't be Ariella, because Ariella is learning the clarinet. We know she can't be Danita, because Danita must be at least 12 in order to be 4 years older than the trumpet player, and the flute player is 10. We know she can't be Britany, because Britany's age must be a multiple of 4. That only leaves Callie as the 10-year-old flute player, and the question is answered. Incidentally, the information is enough to eventually let us determine that Ariella is 14 and learning the clarinet, Britany is 8 and learning the trumpet, and Danita is 12 and learning the saxophone.