WYSE – Academic Challenge
Mathematics Solutions (Sectional) – 2021

1. Ans D: First, let R be the total portfolio value, p be the price per share, and q be the number of shares in the portfolio, all three of which are functions of time and which gives us R = pq. Taking derivatives with respect to time gives us $\frac{dR}{dt} = q\frac{dp}{dt} + p\frac{dq}{dt}$. If we place the values of $p = 50$, $q = 10{,}000$, $\frac{dq}{dt} = 150$, and $\frac{dR}{dt} = -18{,}500$ into the equation, we end up with $-18{,}500 = 10{,}000\frac{dp}{dt} + 50(150)$, which we solve to get $\frac{dp}{dt} = -2.60$.

2. Ans D: Let x feet be the distance from the base of the ladder to the wall, making x + 21 the distance from the ground to the top of the ladder. By the Pythagorean Theorem, $x^2 + (x+21)^2 = 40^2$. We end up with the solutions, $x = -36.763$ and 15.763, but only the "plus" version matters. The angle the ladder makes with the ground is $\tan^{-1}\left(\frac{15.763 + 21}{15.763}\right) \approx 66.8^\circ$.

3. Ans A: By law of cosines, $b^2 = 100^2 + 150^2 - 2(100)(150)\cos(84^\circ)$, so $b \approx 171.36$.

4. Ans A: First note that the focus and the directrix are two units apart. The vertex is halfway between the focus and directrix, in this case, 1 unit away from each.

5. Ans E: The radicand in the numerator has a restriction of $3 - x \geq 0$, which then becomes $x \leq 3$. The denominator has a restriction of $x + 2 \neq 0$, which then becomes $x \neq -2$. Combine these two together to get a domain of $(-\infty, -2) \cup (-2, 3]$.

6. Ans B: Based on a radius of 1.25 ft, height of 6.75 ft, and the formula of $V = \pi r^2 h$, we get $V = \pi(1.25)^2(6.75) \approx 33.13$ ft^3. Converting to gallons gives about 248 gal.

7. Ans B: Rearrange to get $t^2 = x - 3$ (with $x \geq 3$), and substitute to get $y = (x - 3) - 9 = x - 12$, which is a linear ray for $x \geq 3$.

8. Ans A: let x represent the number of \$10,000 kiosk.
$(70 - x)20{,}000 + 10{,}000x = 900{,}000$. Solving for x we have $x = \frac{-500{,}000}{-10{,}000} = 50.$

9. Ans B: Such committees will have 4, 5, or 6 women. So the number we are considering is as follows: C(11, 2)·C(7, 4) + C(11, 1) ·C(7, 5) + C(7, 6) = 2,163.

10. Ans B: Let r be the radius of the circle. This makes the circumference of 1 equal to $2\pi r$, and thus $r = \frac{1}{2\pi}$. The overall equilateral triangle can be split into six equal right triangles with vertices present at the center of the circle, the vertices of the overall triangle, and at the midpoints of the sides of the triangle. Because these right triangles are 30-60-90 triangles, their sides will have lengths of r for the short leg, 2r for the hypotenuse, and $r\sqrt{3}$ for the longer leg. Because six of these component triangle legs make up the perimeter of the overall triangle, that overall perimeter will be $6r\sqrt{3}$. Based on the previously calculated value of r, we get the overall perimeter as $\frac{6\sqrt{3}}{2\pi} \approx 1.65399$.

11. Ans D:
$A = \frac{25}{2}\left[0 + 2(94) + 2(92) + 2(89) + 2(74) + 2(55) + 2(77) + 2(96) + 22\right] = 14{,}700.$

12. Ans C: The dot product of <a, b, c> and <x, y, z> is ax + by + cz. In order for two vectors to be orthogonal, the dot product needs to be 0. The vectors <2, –1, 4> and <0, 0, 0> would qualify under that definition.

13. Ans B: Turn the 4 inches into 1/3 foot, then consider an overall square prism with the playground prism removed, giving us $V = 86 \cdot 86 \cdot \frac{1}{3} - 80 \cdot 80 \cdot \frac{1}{3} = 332$.

14. Ans A: $4x^2 – 0.5y^2 = 1$ is a hyperbola, and the inequality includes (0,0) which is included in the interior.

15. Ans B: Assuming that r and V are functions of t, we have $V = \frac{4}{3}\pi r^3$. This means $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$ and thus $\frac{dr}{dt} = \frac{dV}{dt} \cdot \frac{1}{4\pi r^2}$. When r = 1.5, $\frac{dr}{dt} = 3 \cdot \frac{1}{4\pi(1.5)^2} \approx 0.1061$. The diameter is increasing at double this rate, or about 0.2122.

16. Ans C: First, we are adding 16 – 2 + 1 = 15 terms. Given the sum formula $S_n = \frac{15}{2}[a_1 + a_n]$, we have $a_1 = 5(2) + 2 = 12$, $a_n = 5(16) + 2 = 82$, and a total of $S_n = \frac{15}{2}[12 + 82] = 705$.

17. Ans E: First do a quadratic substitution $3 \cdot \left(7^x\right)^2 + 7 \cdot 7^x + 2 = 0$ so we can factor to get $\left(3 \cdot 7^x + 1\right)\left(7^x + 2\right) = 0$. This gives us $7^x = -\frac{1}{3}$ or $7^x = -2$, neither of which have real solutions.

18. Ans A: If we let x be the number of hours after noon, then solving the equation $80(x-5/60)=75x$ determines when car A catches car B. We get $x=\frac{4}{3}=\frac{80}{60}$ hours after noon, which translates to 1:20 PM. The distance is $\frac{80}{60}\cdot 75=100$ miles.

19. Ans A: $(x-5-2i)(x-5+2i)=(x-5)^2-4i^2=x^2-10x+25+4=x^2-10x+29$

20. Ans A: The cost of \$16,000 corresponds to C = 16. Solving for n gives us 0.375, or 375 items.

21. Ans A: From the complex form we get x = 3 and y = 4 . Then $r=\sqrt{3^2+4^2}=5$, and $\tan\theta=\frac{4}{3}$, which implies that $\theta\approx 53.1°$. This gives us $5\left(\cos(53.1°)+i\sin(53.1°)\right)$.

22. Ans C: The basic absolute value function has a range of $[0,\infty)$. The expression inside the absolute value do not change the range, but the negative in the front changes the range to $(-\infty,0]$, and the addition shifts the range up to $(-\infty,4]$.

23. Ans E: We will use the complimentary event to solve. $1-\text{P(at most 1 yellow)}$.

$$1-(\text{P(0 yellow)}+\text{P(1 yellow)})=1-\left(\frac{{}_{10}C_6}{{}_{17}C_6}+\frac{{}_7C_1\cdot{}_{10}C_5}{{}_{17}C_6}\right)$$

$$=1-\left(\frac{210}{12376}+\frac{7\cdot 252}{12376}\right)=1-\frac{1974}{12376}=\frac{743}{884}$$

24. Ans A: The derivative is $y'=2e^{2t}-\frac{6}{t^2}-5$. If we plug in t = ln2, we get about -9.49.

25. Ans B: The percentage of all birds with gray plumage and blue eyes is 0.20*0.40 = 0.08 or 8%. The percentage of all birds with black plumage and blue eyes is 0.80*0.70 = 0.56 or 56%. This means the total percentage of all birds with blue eyes is 8+56 = 64%. Of this 64%, the percentage of them with gray plumage is 8/64 = 12.5%.

26. Ans D: $2\left(4^{x-1}\right)=15^x\Rightarrow\log 2+(x-1)\log 4=x\log 15\Rightarrow\log 2+x\log 4-\log 4=x\log 15$

$$\Rightarrow x\log 4-x\log 15=\log 4-\log 2\Rightarrow x(\log 4-\log 15)=\log 4-\log 2$$

$$\Rightarrow x=\frac{\log 4-\log 2}{\log 4-\log 15}\approx -0.524$$

27. Ans B: $\frac{\log x}{\log 2}+\frac{\log x}{\log 8}=\frac{3\log x}{3\log 2}+\frac{\log x}{3\log 2}=\frac{4\log x}{3\log 2}=\frac{4\log x}{\log 8}=4\log_8 x=\log_8 x^4$. Note that in base two, the expression is equivalent to $\frac{4}{3}\log_2 x$, and in base 16 it is equivalent to $\frac{16}{3}\log_{16} x$, neither of which can be turned into statements I, III, or IV.

28. Ans E: $0.07(8000) + x(11{,}000) = 1550 \Rightarrow x = 0.09$ or 9%

29. Ans E: First note that sin 3x has a period of $\frac{2\pi}{3} = \frac{4\pi}{6}$ and cos 4x has a period of $\frac{2\pi}{4} = \frac{\pi}{2} = \frac{3\pi}{6}$. Since with sums and differences the period is the least common multiple of that for the two functions involved, the period here is $\frac{12\pi}{6} = 2\pi$.

30. Ans B: To get the right multiples and totals, the four mileages must be 10k, 20k, 30k, and 40k, and the four purchase amounts must be \$5, \$10, \$15, and \$20. Next let's determine the mileage of each car. The Ford was given as 30k. The Honda can't be the least by II or the most by IV, so it must be 20k. That makes the Chevy 10k by II, and the BMW 40k by elimination. For dollar amounts, III tells us the \$15 was spent by the 10k vehicle owner who has the Chevy, and IV tells us the \$10 person must have 30k, which is the Ford owner. The BMW spent more than the 20k Honda owner, so the BMW owner must have spent the \$20, leaving the Honda owner spending the \$5. The two who spent the most were the Chevy and BMW owners, and their 10k and 40k mileages add up to 50k.