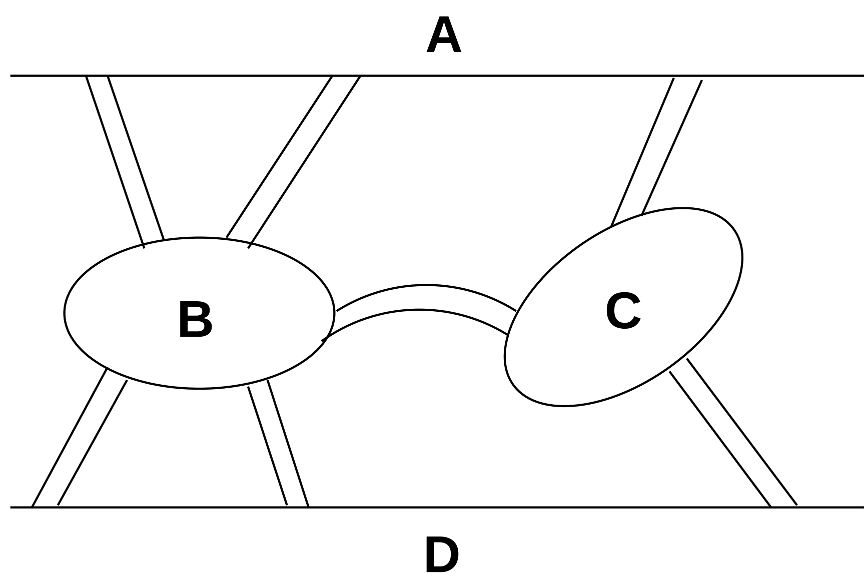
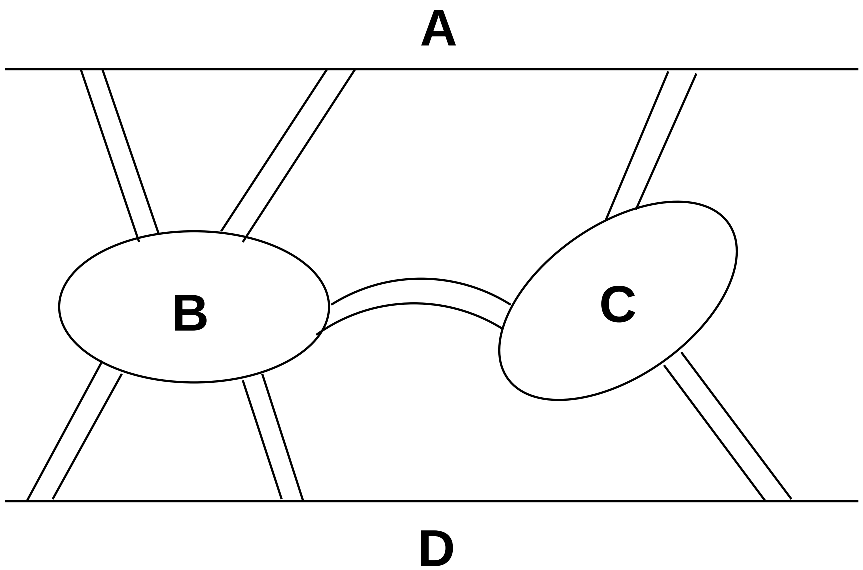
Konigsberg Bridge Problem

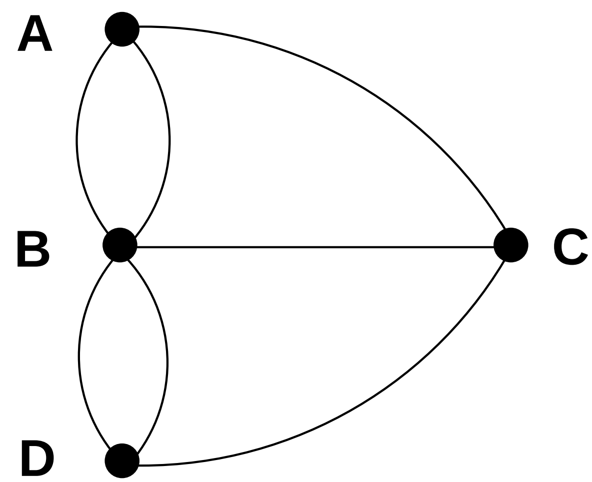
The old Prussian city of Konigsberg, located on the banks of the Pregel River, included two islands which were joined to the banks and to each other by seven bridges:



Because sex and television hadn't been invented yet, the townspeople strolled about the town and across the bridges, and had entirely too much time to think...

Eventually, someone tried to determine a walk which began at their front door, crossed each bridge exactly once, and allowed them to return to their front door…





They weren't able to do this, so took the problem to the famous and fabulously well respected mathematician, Leonhard *"Lenny"* Euler!

He was able to solve the problem, and thus spawned the branch of mathematics known as **Graph Theory**!

Graphs

- A **graph** consists of a set of **vertices** (points) and a set of **edges** (lines) which join these vertices.

- This definition is very different from what is meant when we say we want to **graph** an equation, for example, or the bar graphs we looked at earlier in the semester.

- The **graphs** we'll be discussing are simply convenient diagrams that show the relations or connections between objects in some collection or set.

- This type of graph enables us to communicate and analyze complex information using a visual method, which is often helpful to humans.

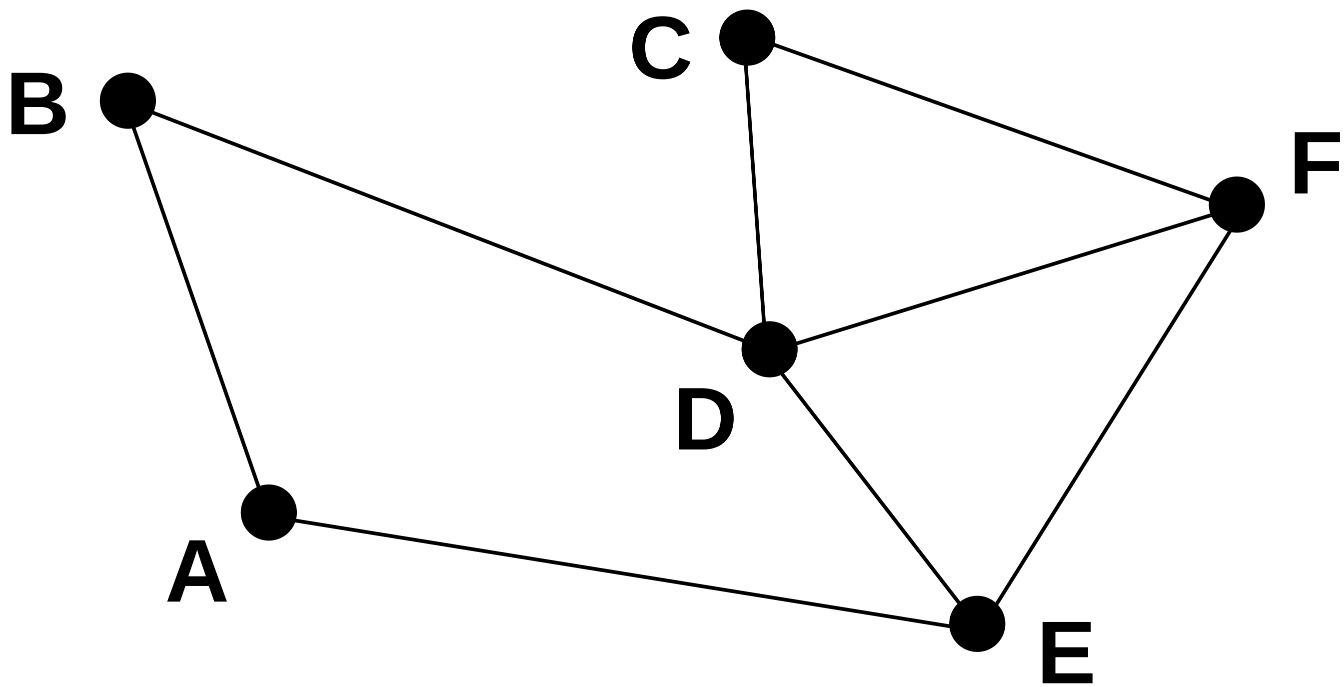
Applications

Graphs have **many** real-world applications, for example:

- road maps and atlases
- chemical molecules
- tournament schedules
- organizational charts
- robotic motion planning
- assembly instructions

You've probably run across many of these examples in your own life.

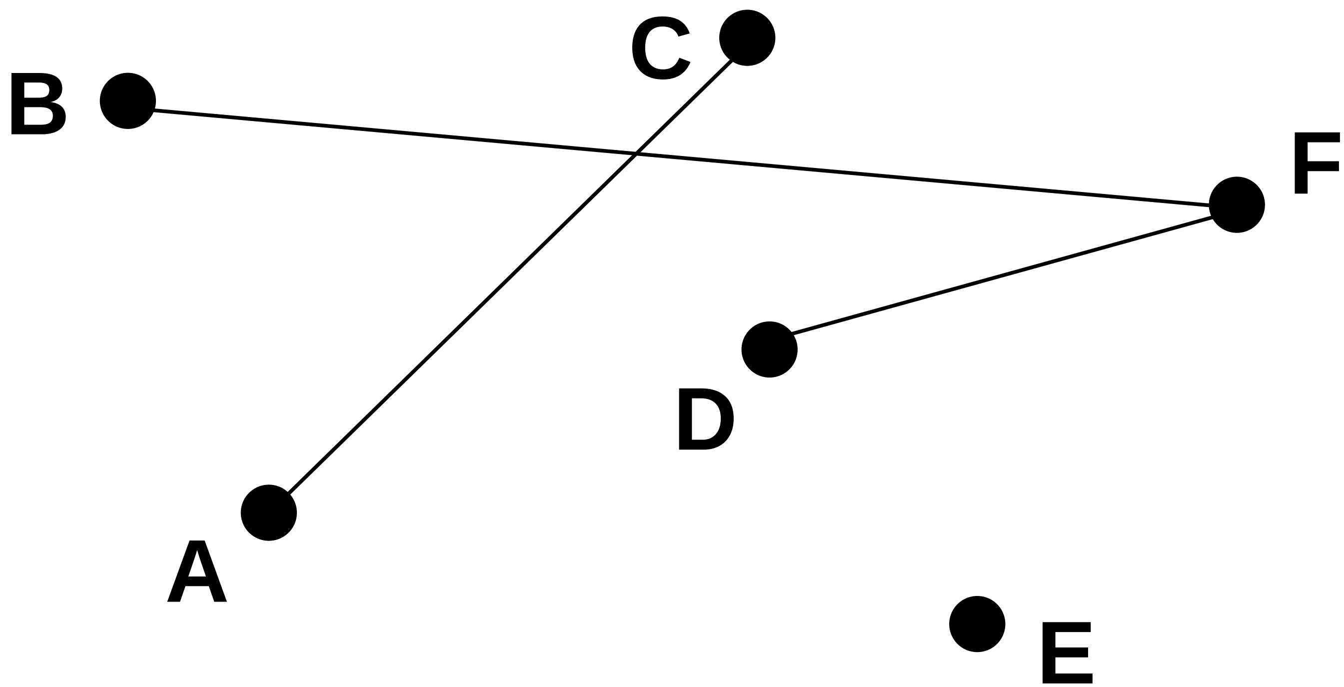
Graph Example — I



Example 1. $G_1 = \{V, E_1\}$

$V = \{A, B, C, D, E, F\}$
$E_1 = \{AB, AE, BD, CD, CF, DE, DF, EF\}$

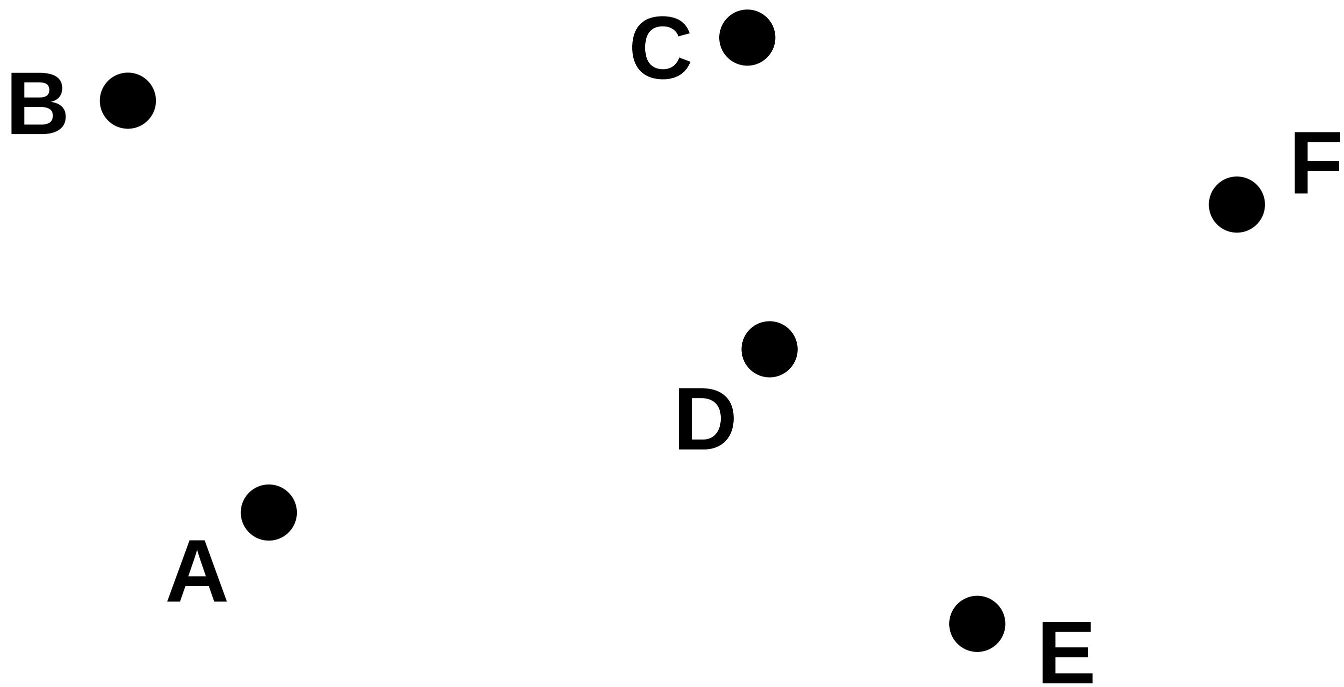
Graph Example — II



Example 2. $G_2 = \{V, E_2\}$

$V = \{A, B, C, D, E, F\}$
$E_2 = \{AC, BF, DF\}$

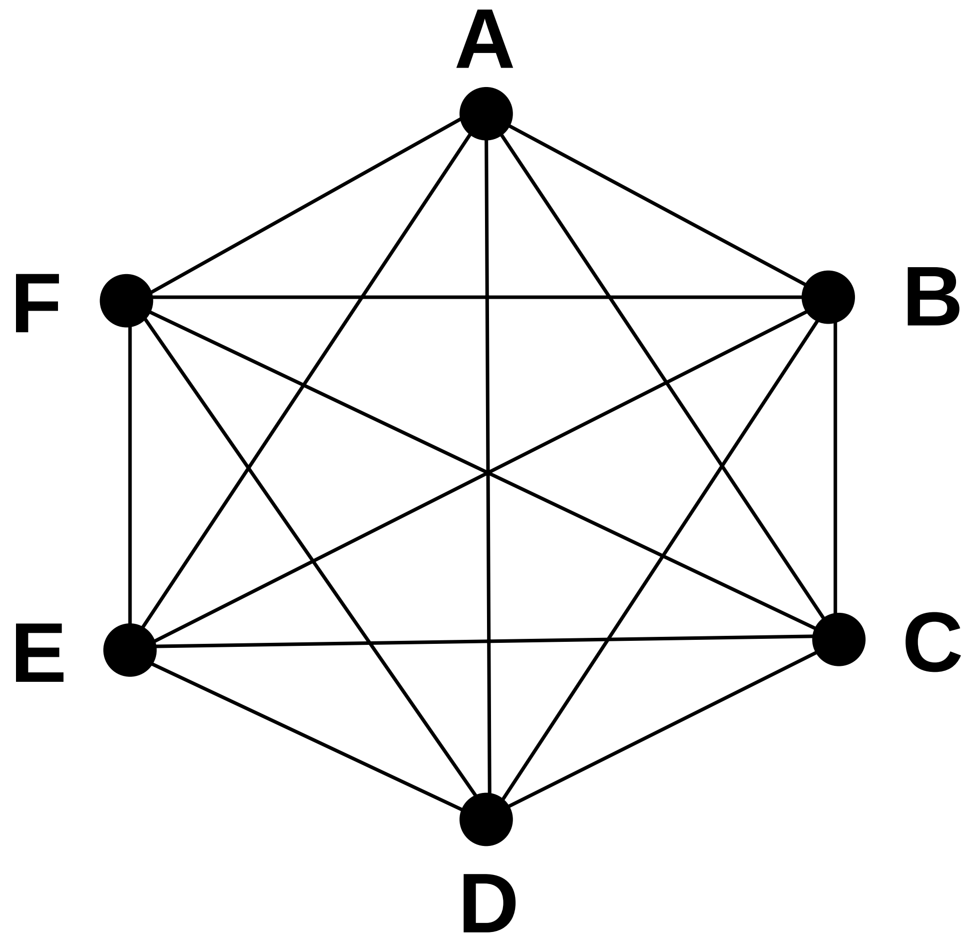
Graph Example — III



Example 3. $G_3 = \{V, E_3\}$

$V = \{A, B, C, D, E, F\}$
$E_3 = \emptyset$

Graph Example — IV



Example 4. $G_4 = \{V, E_1\}$

$V = \{A, B, C, D, E, F\}$

$E_3 = \{AB, AC, AD, AE, AF, BC, BD, BE, BF, CD, CE, CF, DE, DF, EF\}$

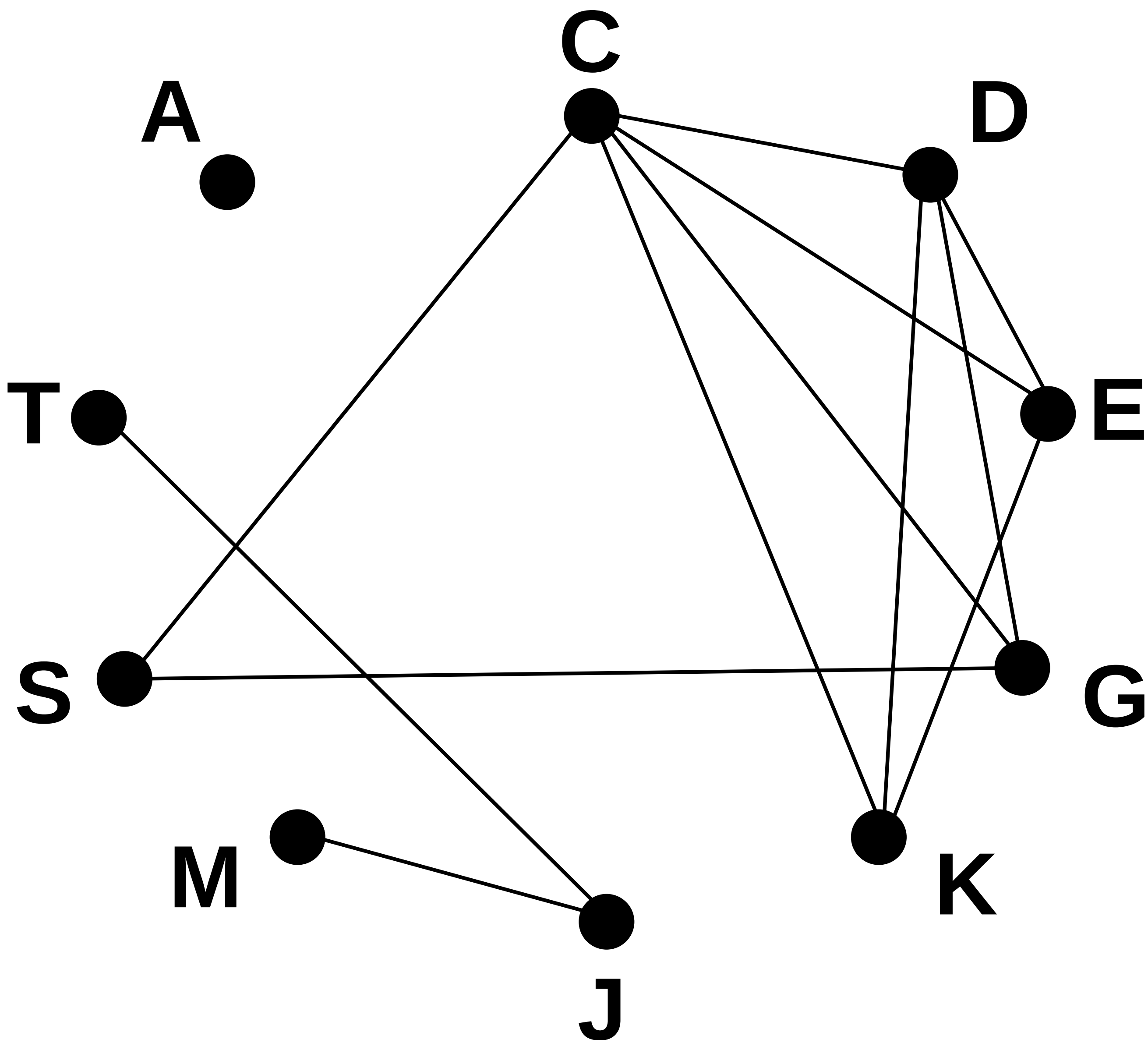
Representing Data

Consider this situation: a health-care worker has 10 clients assigned to her: Andy, Claire, Dave, Erin, Glen, Katy, Joe, Mike, Sam, and Tim. This worker needs to determine which groups of her clients have had or shared social interactions with each other. She knows:

Person	Played with:
Andy	no one
Claire	Dave, Erin, Glen, Katy, Sam
Dave	Claire, Erin, Glen, Katy
Erin	Claire, Dave, Katy
Glen	Claire, Dave, Erin
Katy	Claire, Dave, Erin
Joe	Mike, Tim
Mike	Joe
Sam	Claire, Glen
Tim	Joe

However, even when the information is presented in a table, it is not easy to see the **patterns** in her client's friendships.

Graphical Representation of the Same Data



Notes

- Each letter is the first letter of one of the client names.
- Each person is represented by a dot or **vertex**.
- There are 10 vertices in this graph.
- Two people who socially interact are connected by a line or **edge**.
- There are 12 edges in this graph.
- The vertices at the ends of an edge are called its **endpoints**, and edges must always begin and end at vertices.
- We have a **visual** model or representation of the data, which often helps us see relationships better.

More Notes on Graphs

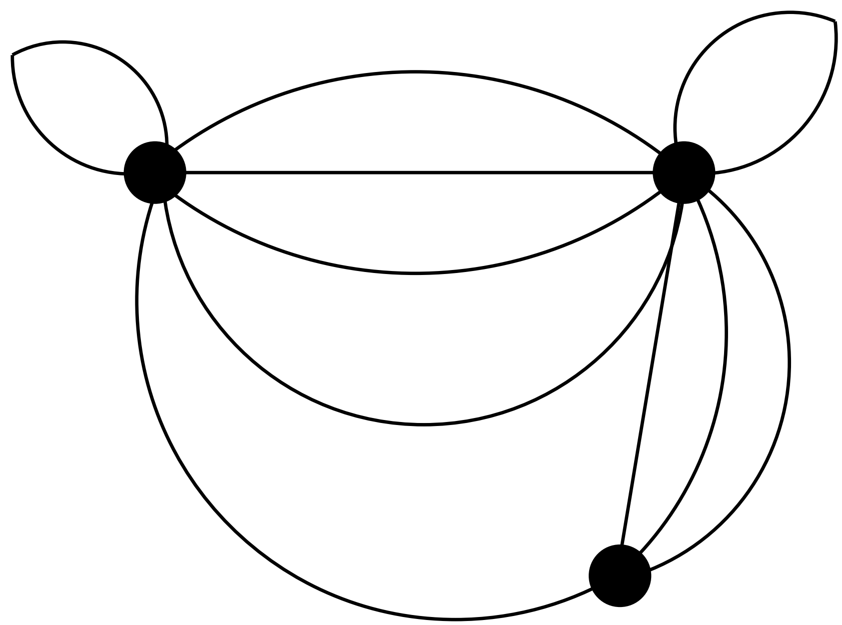
- A **graph** is a set of vertices (at least one) and edges, where each edge connects vertices in the graph.
- Note that **there is no vertex** where edges merely **cross**, as **DK** and **GC** did in the previous graph.
- The **relative positions** of the vertices and the **lengths** of the edges have no significance.
- All that is important is the relationship between the vertices: which are connected by edges, and which are not.
- Edges need not be drawn as straight lines — they may be curved.
- Why not draw **two** edges for each relationship — for example, one to show that Claire plays with Sam, and another to show that Sam plays with Claire?

  Extra edges wouldn't show any additional information in this case, and would needlessly complicate the graph.

Simple and Multi-Graphs

- Graphs which restrict the number of edges between any two vertices to **one** are called **simple graphs**. **Unless indicated otherwise, assume the graphs we discuss are simple graphs.**

- Graphs which allow multiple edges between the same vertices, as well as permitting self-loops, are called **multi-graphs**.



Important Definitions

- Two vertices are **adjacent** if they share an edge.

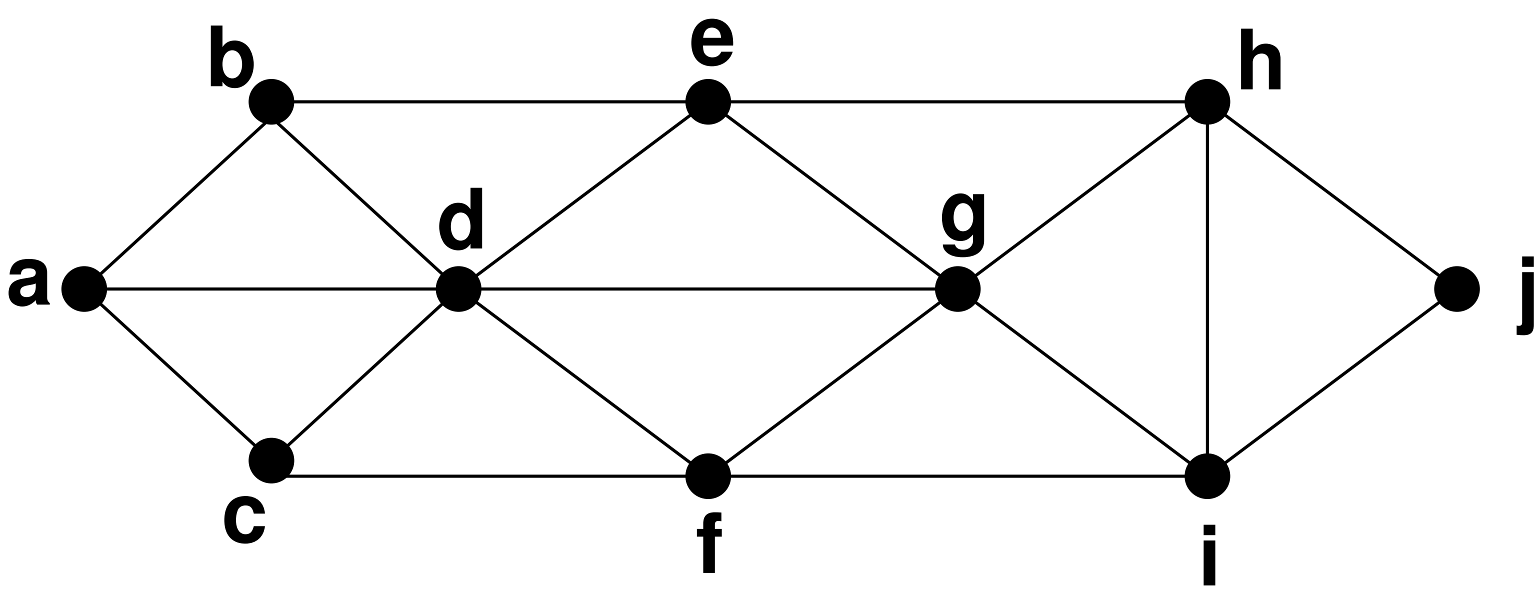
- An edge is **incident** to the two vertices which are its endpoints (and not to any others).

- The **degree of a vertex** is the number of edges which are incident to it.

- We can **add the degrees** of all the vertices and **divide by two** to determine the **number of edges** in a graph.

Degrees of Vertices

- What are the degrees of the vertices in the graph below?

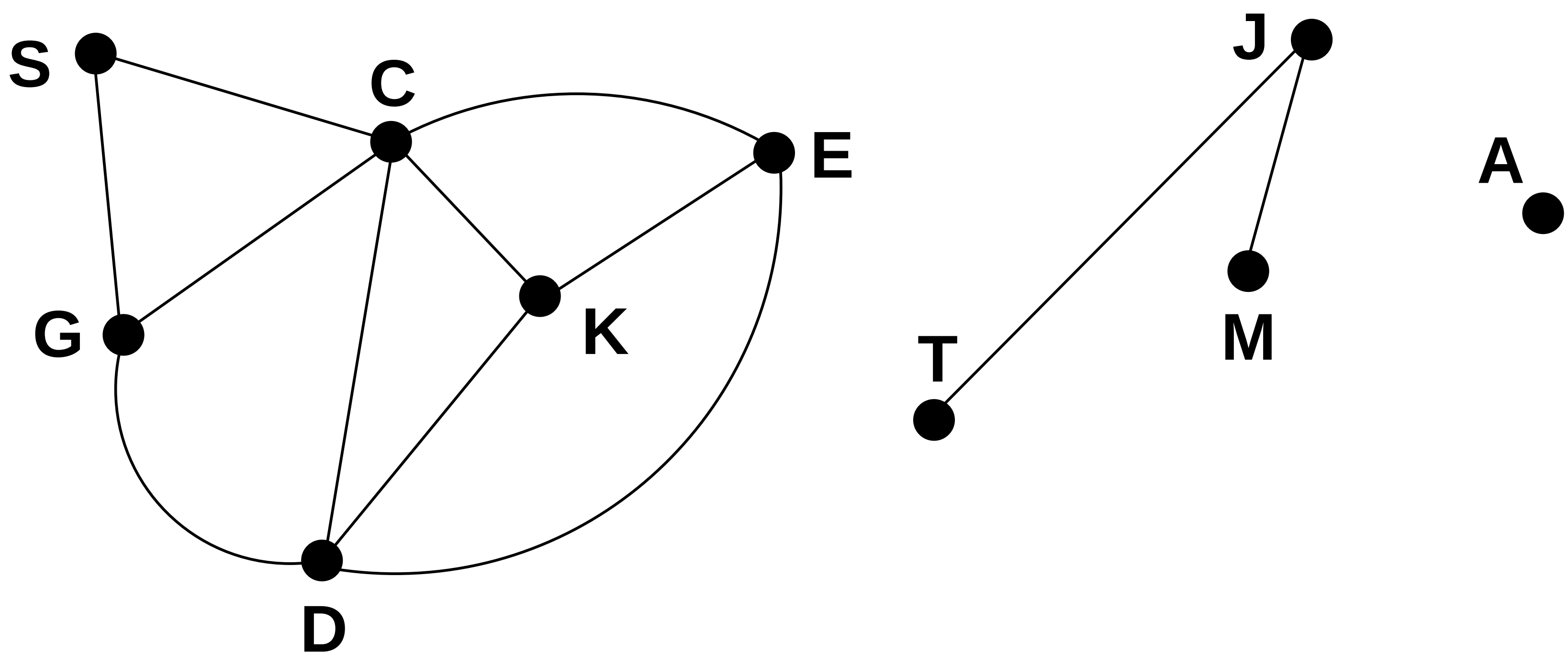


a ______ b ______ c ______ d ______ e ______

f ______ g ______ h ______ i ______ j ______

Vertex Placement

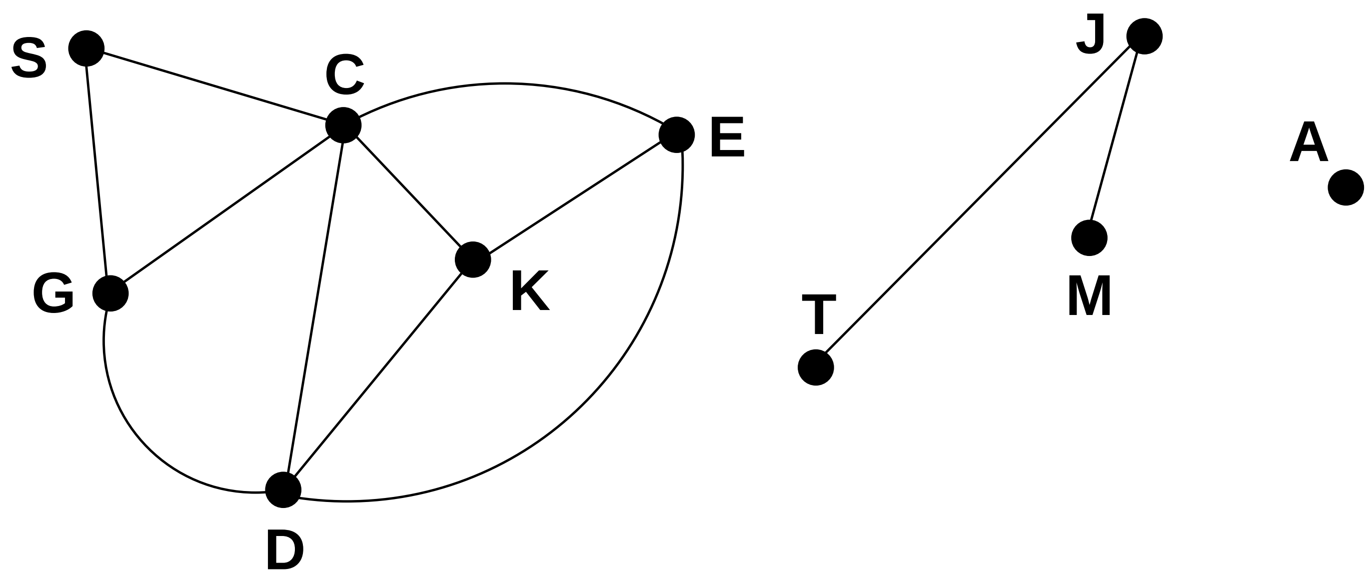
The graph below may look different from the graph depicting the friendship patterns earlier, but notice that although the vertices are in different places, the graph still has edges between the same pairs of vertices:



Walks and Paths

- A **walk** in a graph is a **sequence of vertices**, each linked to the next vertex by a specified edge of the graph.

- A **path** in a graph is a **walk** that uses no edge more than once.

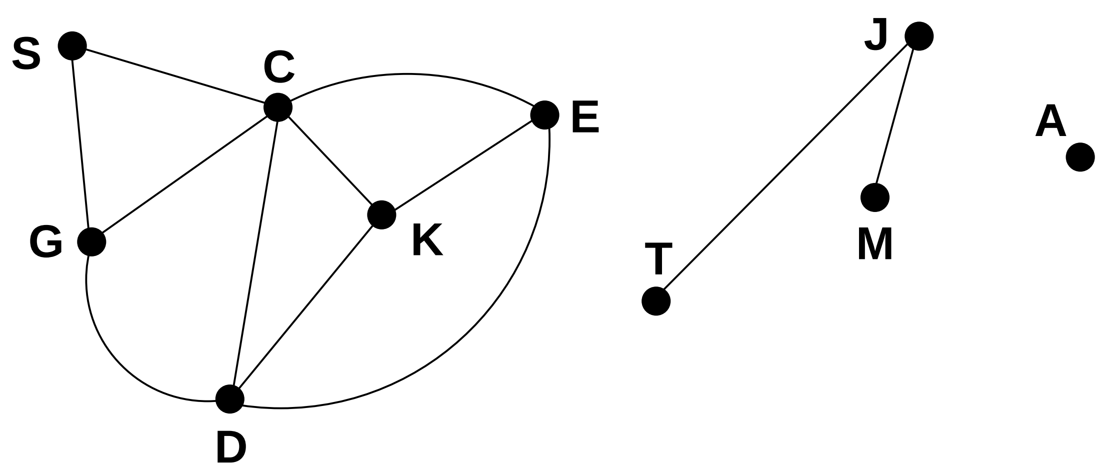


Walks:

Paths:

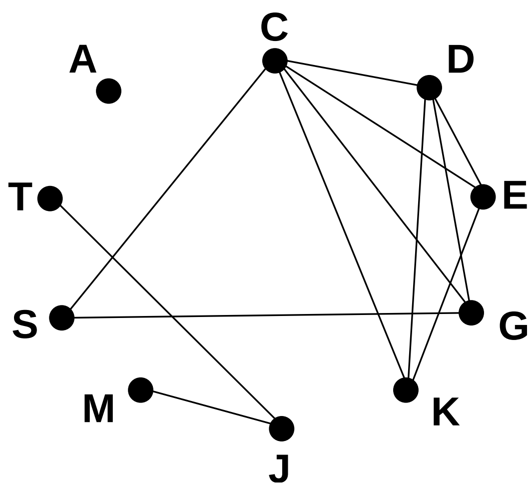
Connectedness and Subgraphs

- A graph is **connected** if every pair of vertices in the graph is connected by a path.
- A **subgraph** of a graph consists of a non-empty subset of the vertices of the graph and 0 or more of the edges between those vertices. (Edges which are adjacent to vertices outside the subset are not allowed).
- The **connected components** of a graph are the collection of those subgraphs containing all "reachable" vertices and their edges (and thus which are connected).

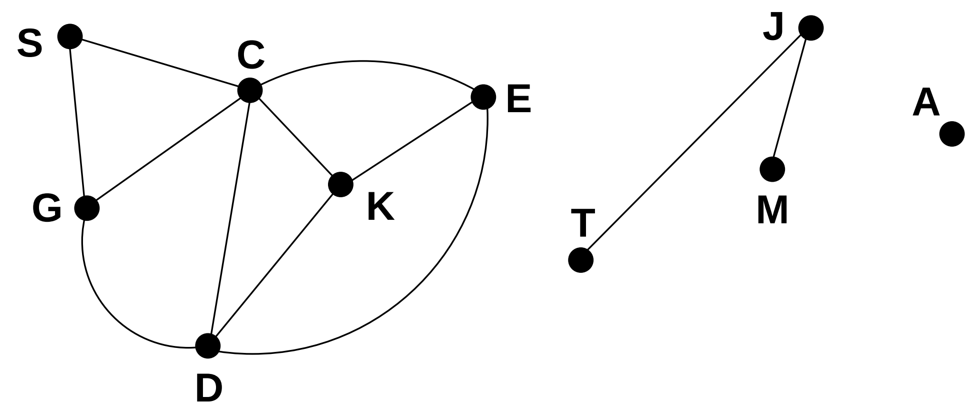


Graph Isomorphism

We say the two graphs are **isomorphic** to one another:



Graph I

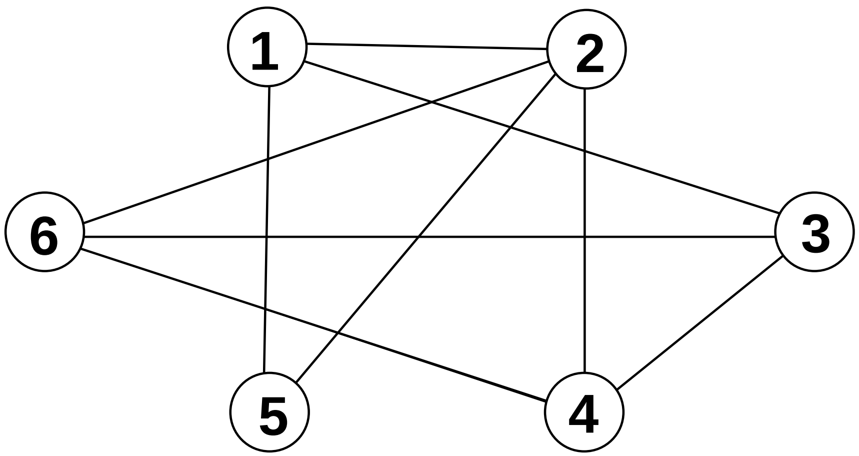


Graph II

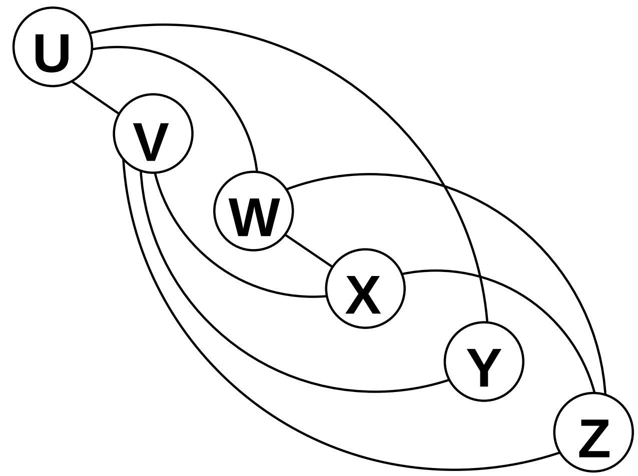
- Two graphs are **isomorphic** if there is a **one-to-one** correspondence between vertices of the two graphs with the property that whenever there is an edge between two vertices of either one of the graphs, there is an edge between the corresponding vertices of the other graph.

Isomorphism

- Two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ are said to be **isomorphic** if there exists a one-to-one onto mapping $f : V_G \rightarrow V_H \ni$
$$< u, w > \in E_G \text{ IFF } < f(u), f(w) > \in E_H$$
- I.e., we can relabel the vertices of G to be vertices of H, maintaining the corresponding edges in G and H;
pairs are adjacent in G IFF pairs are adjacent in H



Graph G



Graph H

- The mapping from V_G to V_H given by

```
f(1) = u,  f(2) = v,  f(3) = w,
f(4) = x,  f(5) = y,  f(6) = z
```

is the requisite mapping.

More on Isomorphism

- If we can find a **matching** between the vertices of two graphs such that whenever there is an edge between vertices in one graph, the corresponding vertices also share an edge, then the graphs are **isomorphic**.

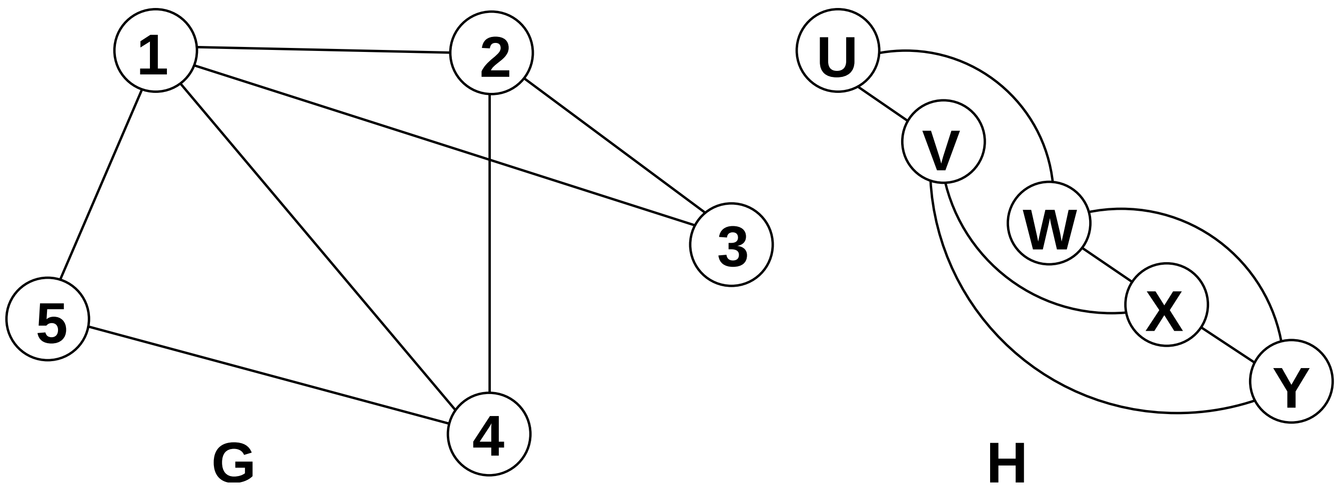
  - Suppose vertex A in Graph1 maps to vertex Z in Graph2, and vertex B in Graph1 maps to vertex Y in Graph2.

  - Then if the edge AB exists in Graph1, the edge ZY exists in Graph2, and vice-versa

  - If there is no edge AB, then there is no edge ZY, and vice-versa

Non-Isomorphic Graphs

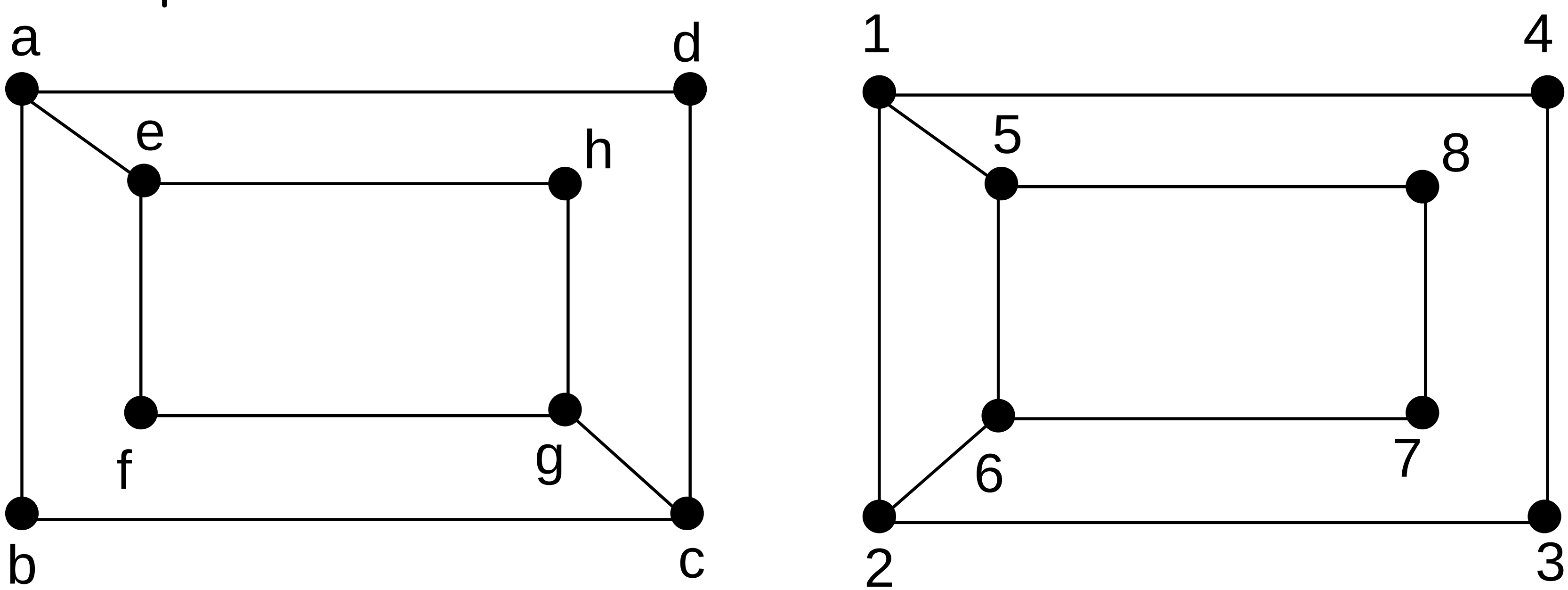
- These two graphs are **not** isomorphic since deg(1) = 4, and no vertex in graph H has degree 4.



- Note: degrees are preserved under isomorphism

Isomorphic Subgraphs

- If we cannot find isomorphic subgraphs, then the graphs are not isomorphic.



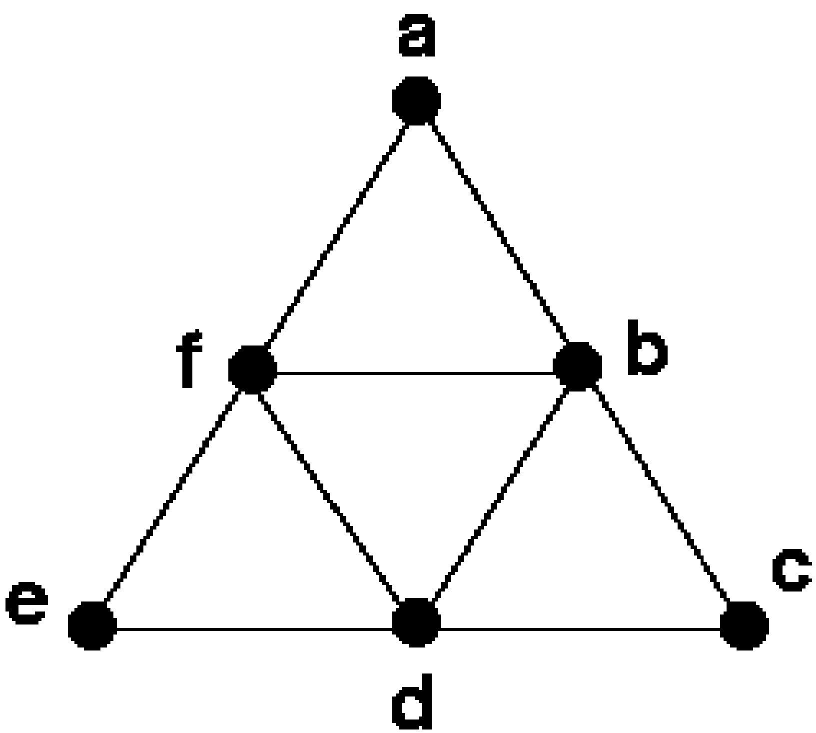
deg 2:	**b,d,f,h**	**3,4,8,7**
deg 3:	**a,c,e,g**	**1,2,5,6**

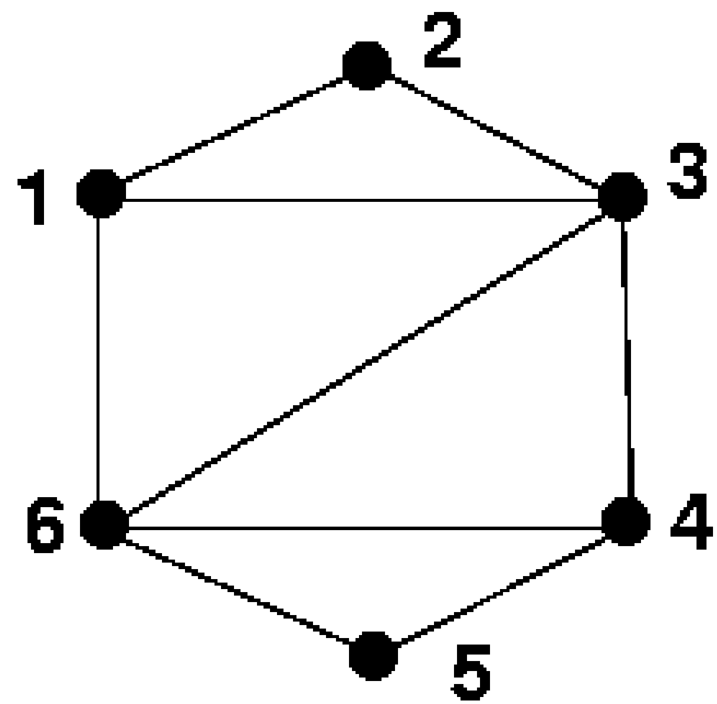
- Subgraphs containing these (deg 2) vertices must be isomorphic.
- No edges between **b,d,f,** or **h** (within same set), while edges **<3,4>** and **<7,8>** exist. Therefore the two graphs are not isomorphic.

Requirements for Two Graphs to Be Isomorphic

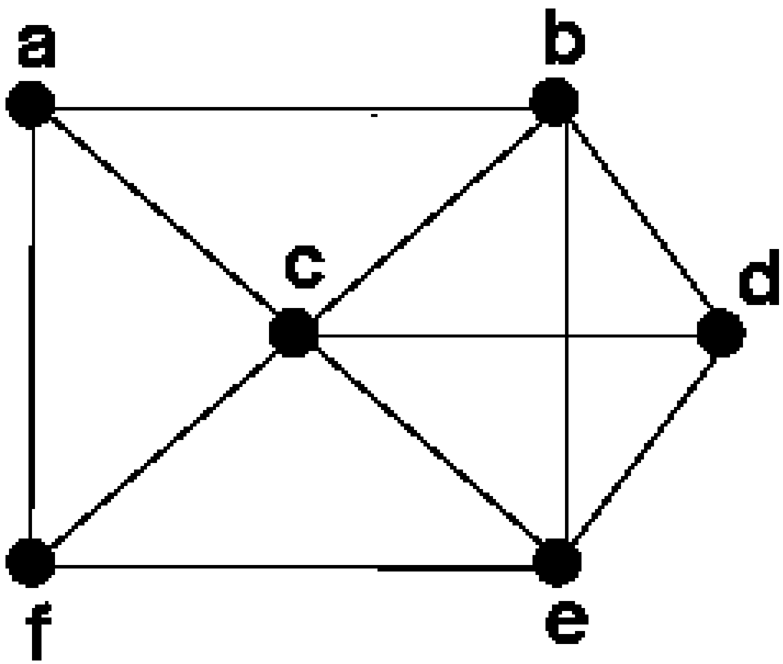
- Isomorphic graphs must have the **same**:
  - number of components
  - number of vertices
  - degrees of vertices
  - number of edges
  - **subgraphs** — vertices with the same degrees in the same arrangements

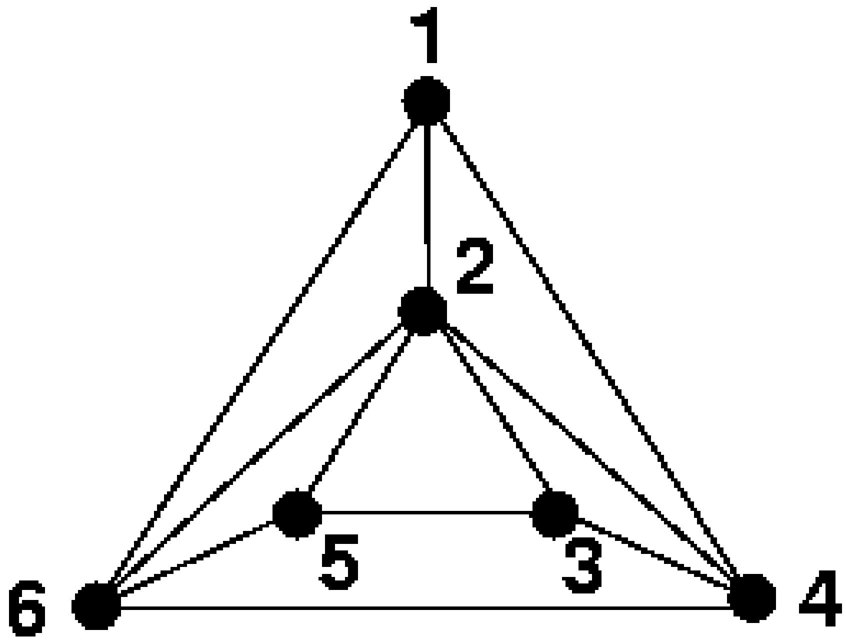
Are These Graphs Isomorphic?



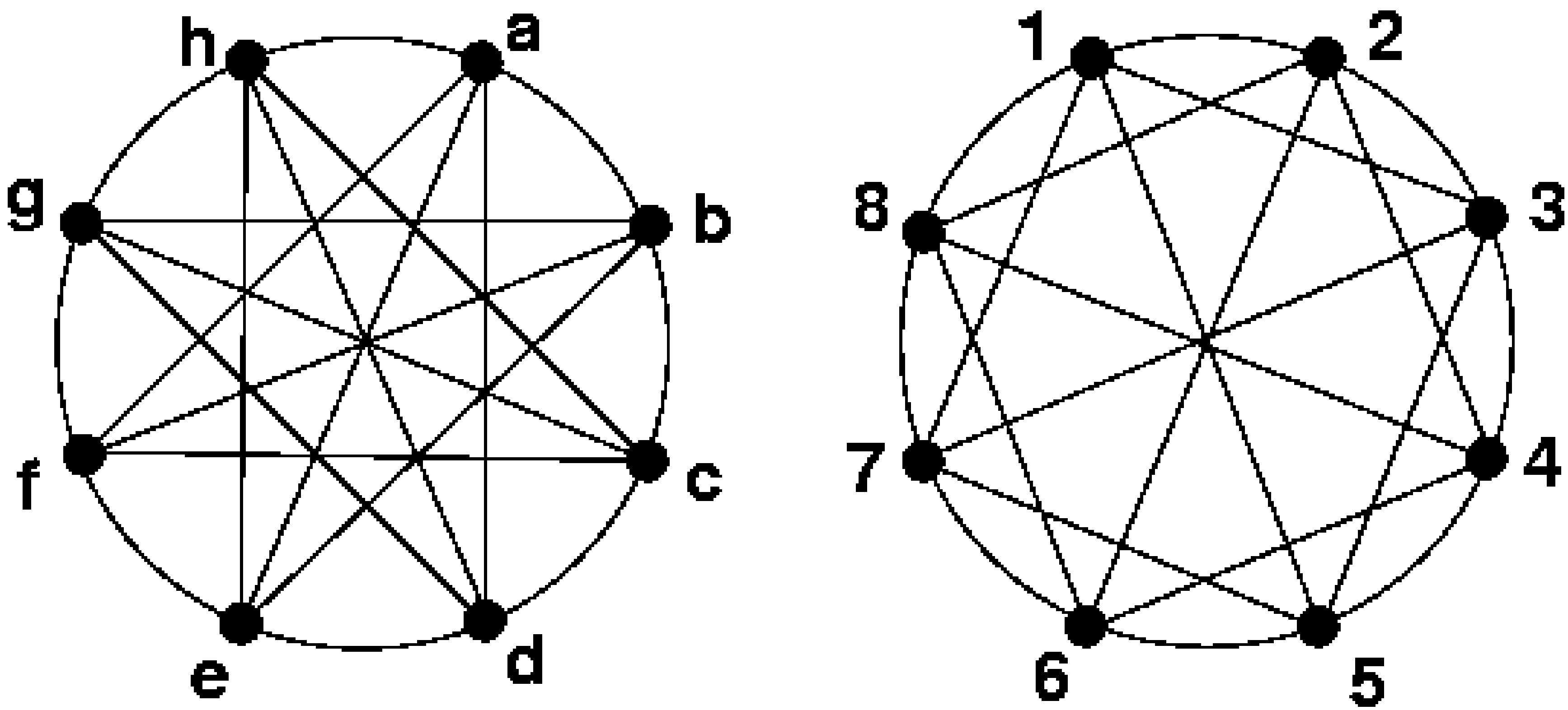


Are These Graphs Isomorphic?

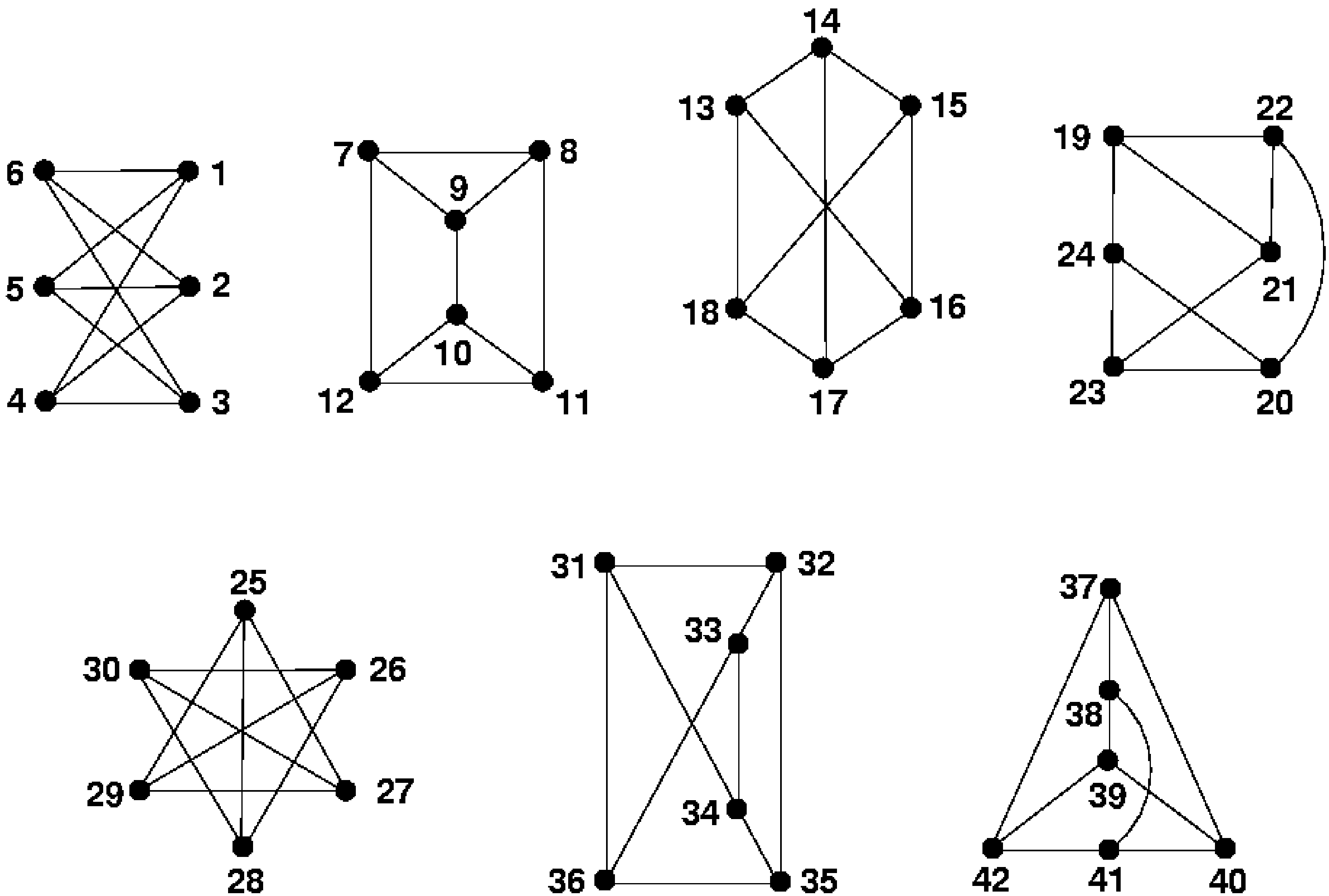




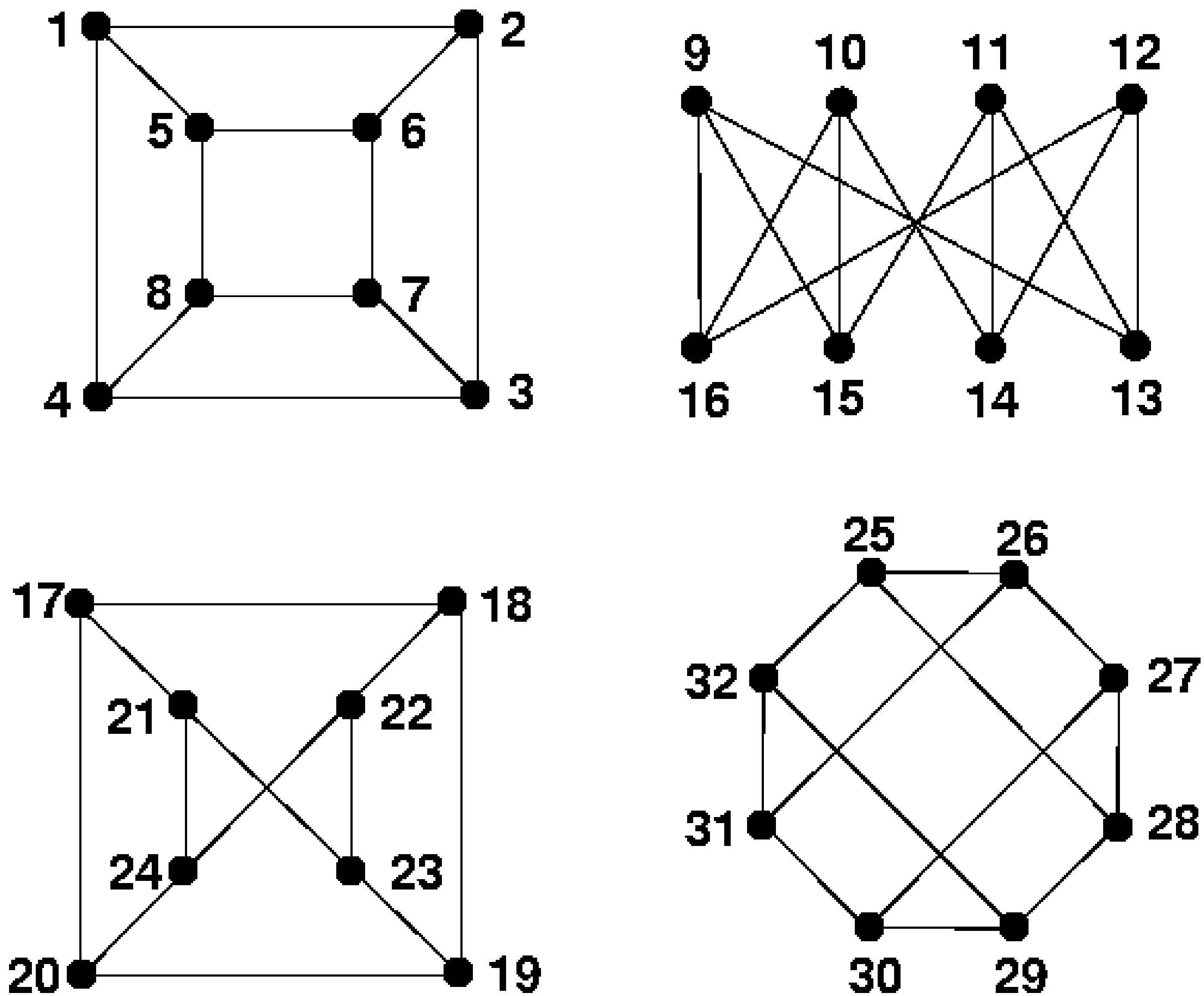
Are These Graphs Isomorphic?



Which of These Graphs Are Isomorphic?



Which of These Graphs Are Isomorphic?



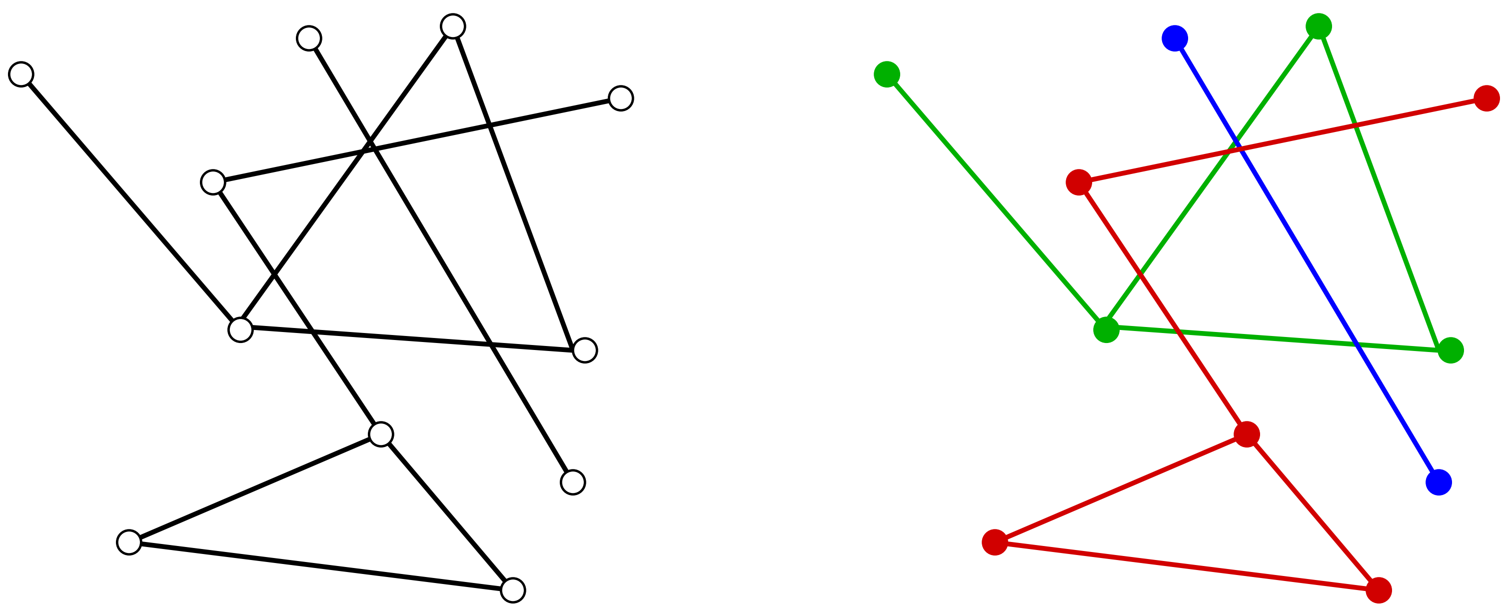
More on Graphs

A graph is **connected** if one can move from each vertex of the graph to every other vertex of the graph *along edges of the graph*.

If a graph is **not connected**, it is said to be **disconnected**.

The connected pieces of a graph are called the **components** of the graph.

It is helpful to use colors to determine the components of a graph:



Edges And Degrees

Theorem. In any graph, the sum of the degrees of the vertices equals twice the number of edges.

Example. A graph has precisely six vertices, each of degree 3. How many edges does this graph have?

If a graph has 2 vertices, what is the **maximum** number of edges it can have?

If it has 3 vertices?

4 vertices?

n vertices? (This would be a K_n or complete graph over n vertices)

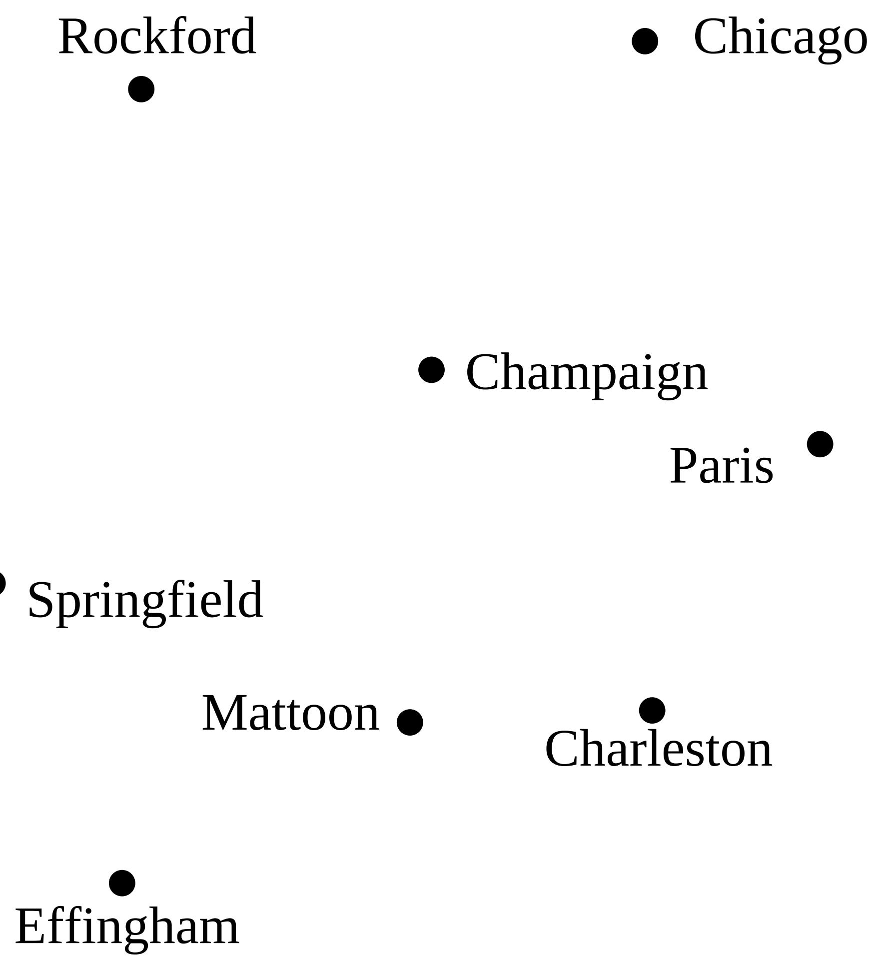
Bus Routes

Example. Suppose Greyhound bus provides direct bus links between the following Illinois cities:

City	Direct bus links with:
Rockford	Chicago, Springfield
Chicago	Rockford, Champaign
Champaign	Chicago, Mattoon, Springfield
Paris	Charleston
Springfield	Rockford, Champaign
Mattoon	Champaign, Charleston, Effingham
Charleston	Mattoon, Effingham, Paris
Effingham	Mattoon, Charleston

Associated Graph

We can represent this information with a graph:



Is this graph connected? What is the real-world significance in this case?

Walks in Graphs

What trips could we take among the destinations, using only direct bus links? **One possibility**: Charleston–Mattoon–Champaign–Chicago
Other trips:

If we don't mind riding the same bus route more than once, what trips could we take?

A **walk** in a graph is a sequence of vertices, each linked to the next vertex by a specified edge of the graph.

We can think of a walk as a route we can trace with a pencil **without lifting it from the graph**.

Paths in Graphs

A **path** in a graph is a walk that **uses no edge more than once**.

A path is a special kind of walk in which we **don't traverse the same edge more than one time**.

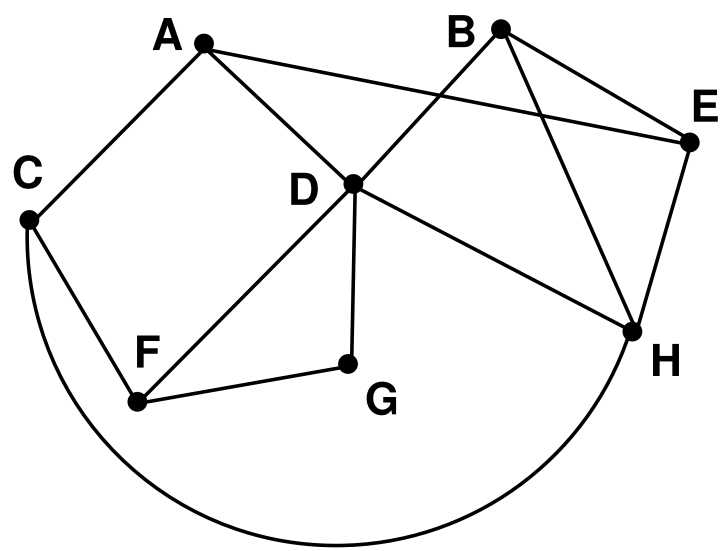
Which of our trips were *paths*?

Circuits

A **circuit** in a graph is a path that begins and ends at the **same vertex**.

A circuit is a kind of path, so therefore it is also a type of walk.

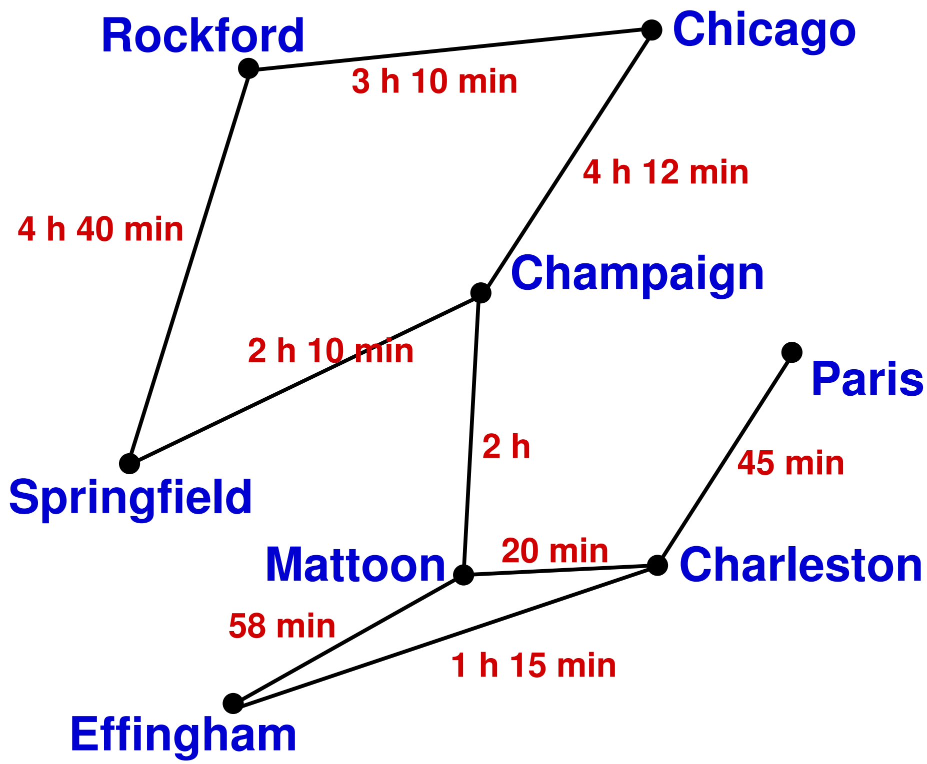
Example. Using the graph below, classify each of the following sequences as a WALK, a PATH, or a CIRCUIT.



Sequence	**Walk?**	**Path?**	**Circuit?**
D-G-F-D			
A-D-G-F-C			
A-B-E-H-G-F-C-A			
A-D-H-E-B-D-A			
D-H-E-B-D			
C-H-E-B-H-D-G-F-C			

Weighted Graphs

A graph with numbers on the edges (as shown below) is known as a **weighted** graph. The numbers along the edges are called **weights**.



This graph represents the length of time the bus ride takes between connected cities. It would be useful to know this when attempting to plan a trip.

Total Trip Time

How long would it take to go from Charleston to Mattoon to Effingham?

From Charleston to Chicago?

From Paris to Chicago?

Round trip from Springfield to Charleston and back?

Tournaments

Example. In a round-robin tournament, every team plays every other team. The winners are the team who wins the most games. Suppose six teams compete in such a tournament. How many matches will be played all together? [Draw a sketch]

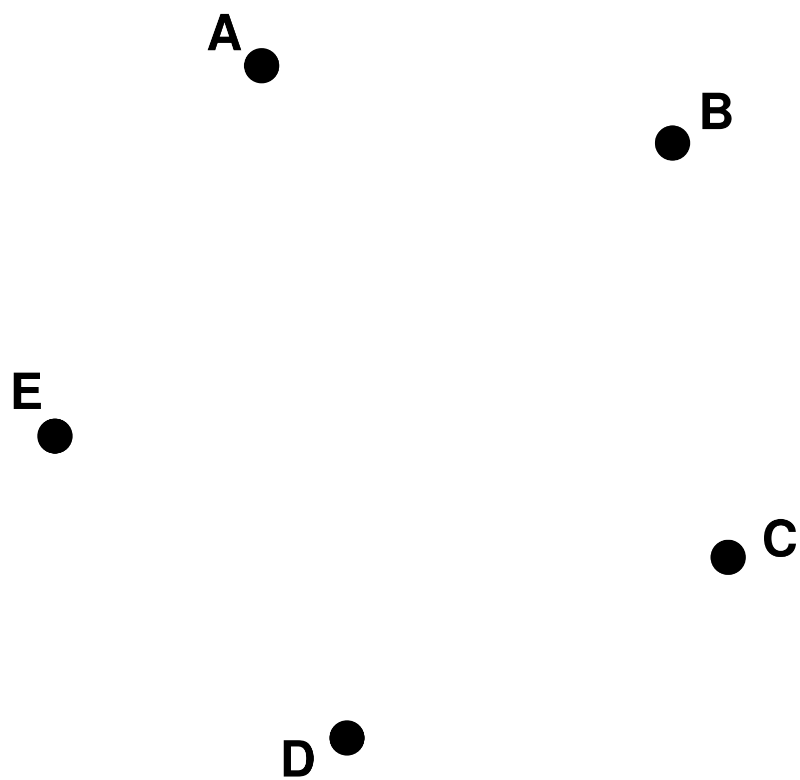
Now, Suppose that these teams play a double-elimination tournament. How many matches will be played?

Complete Graphs

A **complete graph** is a graph in which there is exactly one edge going from each vertex to every other vertex in the graph.

Find a complete graph of 3 vertices, a K_3, in the previous map.

Draw a K_5, a complete graph over these 5 vertices:



Complete Graphs — Degrees of Vertices

What is the degree of every vertex in a complete graph with:

3 vertices?

4 vertices?

5 vertices?

n vertices?

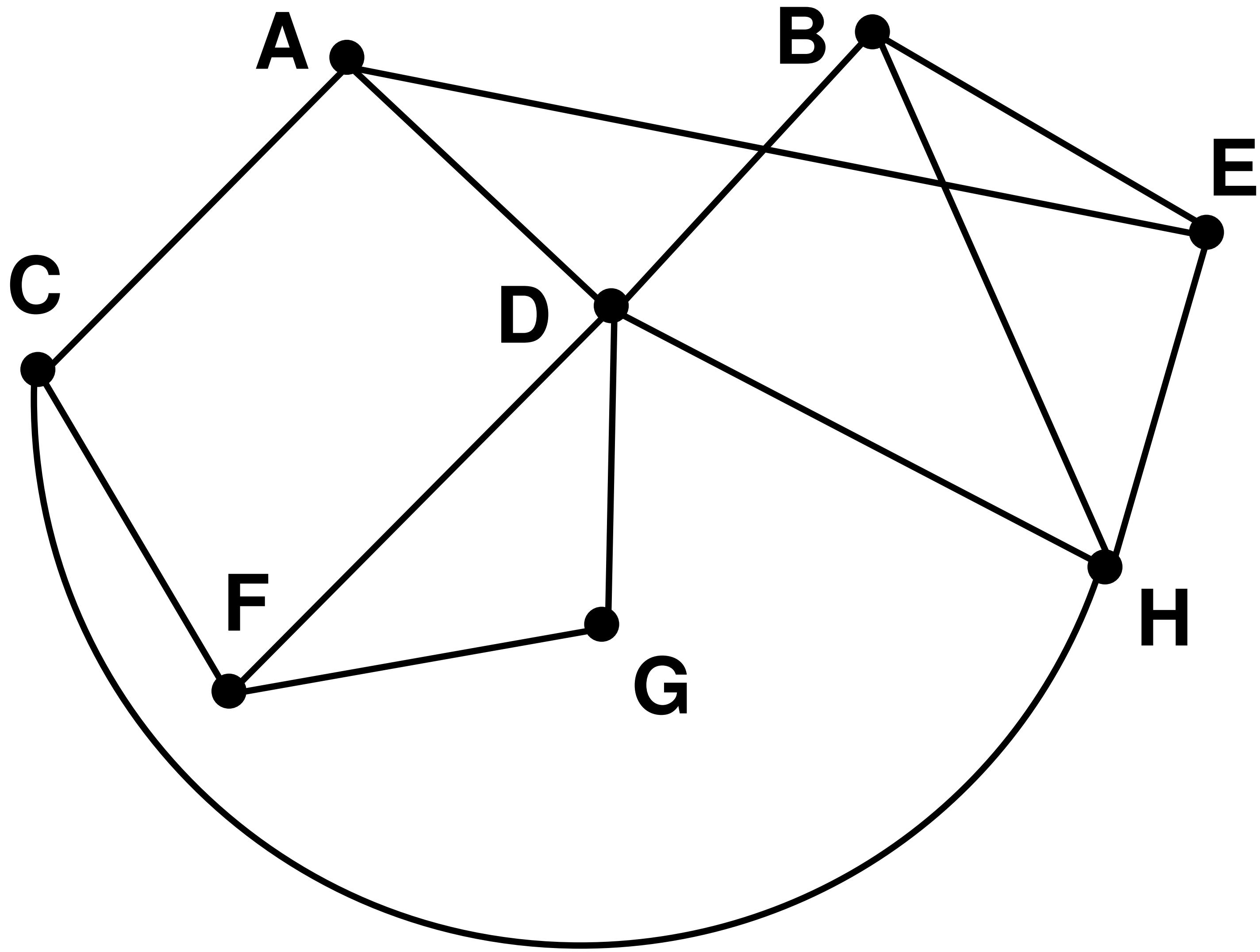
Subgraphs

In general, a graph consisting of **some of the vertices** of the original graph, and **zero or more of the original edges** between those vertices, is known as a **subgraph**.

Vertices or edges not included in the original graph cannot be in the subgraph.

A subgraph may include anywhere from one to all the vertices of the original graph, and may include anywhere from none to all of the edges of the original graph.

Find some subgraphs in this graph



Find A Subgraph. . .

1. with the fewest vertices
2. with the fewest edges
3. with the most vertices
4. with the most edges (the largest subgraph)
5. that's a K_3
6. that consists of two K_3's
7. that consists of three K_3's
8. that is not connected
9. that has three components