Mat 1160
WEEK 6

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Spring 2010

Student Responsibilities – Week 6

- **Reading**:
  This week: Textbook, Section 2.5: Infinite Sets
  Next week: Textbook, Sections 3.1–3.2: Logic and Truth Tables
- Summarize Sections & Work Examples
  Sample Sec 1.1 and 1.2 are on our web site
- Attendance
- Recommended exercises:
  - Section 2.5: (all 1–6), evens 8–46
  - Chapter test

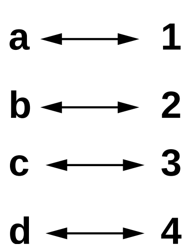
Sec 2.5 Infinite Sets & Their Cardinalities

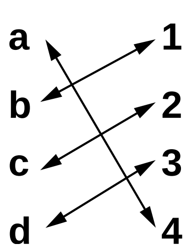
- **Cardinal Number** or **Cardinality**: of a finite set is the number of elements that it contains.
- **One–to–one (1–1) Correspondence**: the elements in two sets can be matched together in such a way that each element is paired with exactly one unique element from the other set.

One–to–one Correspondence Example

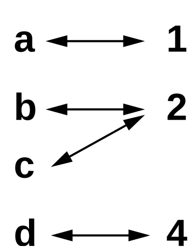
For example, we can find a 1–1 correspondence between the sets { a, b, c, d } and { 1, 2, 3, 4 } by pairing **(a,1), (b,2), (c,3), and (d,4)**.

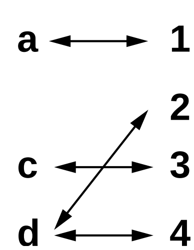
Other matches would work also: **(1,b), (2,c), (3,d), and (4,a)**





- Examples of correspondences which are **not** 1–1:





Set Equivalence

- Two sets A and B are **equivalent**, ($A \sim B$), if they can be placed in a 1–1 correspondence.
- $\aleph_0$, **Aleph–naught** or **Aleph–null**, is the **cardinality** of the natural (or counting) numbers, $\{1, 2, 3, \ldots\}$, a **countably infinite** set.
- If we can show a 1–1 correspondence between some set, A, and the natural numbers, we say that A also has cardinality $\aleph_0$.
- **Thus, to show a set A has cardinality $\aleph_0$, we need to find a 1–1 correspondence between A and the set of natural numbers, $\mathbb{N}$**

- Can we show $|\mathbb{W}| = \aleph_0$? **Yes**:

$$\begin{array}{ccccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \ldots & \mathbf{n} & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \\ \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \ldots & \mathbf{n-1} & \ldots \end{array}$$

- Notice that we just proved that $\aleph_0 = 1 + \aleph_0$!

That is just too weird!

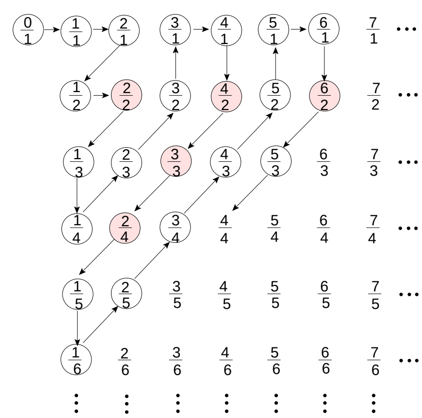
Infinite Sets

- A set is **infinite** if it can be placed in a one–to–one correspondence with a **proper** subset of **itself**.
- Can we show that the set of integers has cardinality $\aleph_0$? Yes:

$$\begin{array}{cccccccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{7} & \ldots\ \mathbf{2n} & \mathbf{2n+1} & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ \mathbf{0} & \mathbf{1} & \mathbf{-1} & \mathbf{2} & \mathbf{-2} & \mathbf{3} & \mathbf{-3} & \ldots\ \mathbf{n} & \mathbf{-n} & \ldots \end{array}$$

- **Good Grief!** We just showed that $\aleph_0 = 2 \times \aleph_0$! The cardinality of the counting numbers is the <u>**same**</u> as the cardinality of the integers!
- There are just as many counting numbers as there are integers...

We can even show that the set of **rational numbers** has cardinality $\aleph_0$.



The correspondence begins...

$$\begin{array}{ccccccc} \frac{0}{1} & \leftrightarrow & 1 & \quad & \frac{1}{1} & \leftrightarrow & 2 \\ \frac{2}{1} & \leftrightarrow & 3 & & \frac{1}{2} & \leftrightarrow & 4 \\ \frac{1}{3} & \leftrightarrow & 5 & & \frac{1}{4} & \leftrightarrow & 6 \end{array}$$

- Numbers that are shaded are omitted since they can be reduced to lower terms, and were thus included earlier in the listing. (This is actually optional).
- The mapping of the positive Rational numbers to the natural numbers is a 1–1 correspondence, so it shows that the **positive Rational numbers have cardinality $\aleph_0$**.
- By using the method in the example for the integers, we can extend this correspondence to include negative rational numbers.
- Thus, **the set of rational numbers, $\mathbb{Q}$, has cardinality $\aleph_0$**.

- A set is **countable** if it is **finite** or if it has **cardinality** $\aleph_0$.
- If a set is both **countable** and **infinite**, we call it **countably infinite**.

- Recall: $\Re = \{ x \mid x \text{ is a number that can be written as a decimal} \}$

- We can show that the set of all real numbers, $\Re$, does **not** have cardinality $\aleph_0$ (and in fact, is larger than $\aleph_0$) using a technique called **diagonalization**.

We will **assume** $|\Re| = \aleph_0$
and
show this leads to a **contradiction**.

- Since we **assumed** $\Re$ **is countably infinite**, there is a **1–1 correspondence between it and** $\mathbb{N}$ (by definition).

Thus some decimal number corresponds to the counting number 1, another to 2, and so on. Suppose:

$$\begin{array}{ccl} 1 & \leftrightarrow & .\mathbf{1}396875\ldots \\ 2 & \leftrightarrow & .4\mathbf{8}13863\ldots \\ 3 & \leftrightarrow & .75\mathbf{2}7790\ldots \\ 4 & \leftrightarrow & .394\mathbf{0}355\ldots \\ 5 & \leftrightarrow & \ldots \end{array}$$

and on and on. . . .

Next we **construct** a decimal number which **cannot** be a part of this correspondence.

But we assumed $|\Re| = \aleph_0$, so **every** decimal number **must appear** somewhere in the correspondence list!

A Contradiction!

Let us construct a decimal number, D, according to the following:

1. The first decimal number in the given list has **1** as its first digit; let D start as $D = .2\ldots$ Thus D **cannot be the first number in the list**.
2. The second decimal in the list above has **8** as its second digit; let $D = .29\ldots$ Thus D **cannot be the second number in the list**.
3. The third decimal in the list has **2** as its third digit; let $D = .293\ldots$ Thus D **cannot be the third number on the list**.
4. The fourth digit of the fourth decimal is **0**, so let $D = .2931\ldots$
5. Continue building D in this manner

Now we get to ask: **Is** D **in the list**
(that we assumed contained **all** decimals numbers?)

NO, since every decimal number in the list differs from D in at least one position, D **cannot possibly be in the list**!! ($\Rightarrow\Leftarrow$)

Note:

1. We assumed **every** decimal number was in the list.
2. But the decimal number D is **not** in the list.

This presents us with a `Contradiction` — i.e.,
these statements **cannot both be true**.

Thus our original assumption was **incorrect**:

It is not possible
to find a 1–1 correspondence between $\Re$ **and** $\mathbb{N}$

.

- The set of real numbers $\Re$ has cardinality $\mathfrak{c}$ (cardinality of the continuum), which is **larger** than $\aleph_0$.

- The Power Set of $\Re$ has cardinality **even greater** than $\mathfrak{c}$
. . . and it only gets worse from there.

- $\mathbb{N}, \mathbb{W}, \mathbb{I}$, and $\mathbb{Q}$ all have cardinality $\aleph_0$

- Irrational numbers and $\Re$ have cardinality $\mathfrak{c}$.